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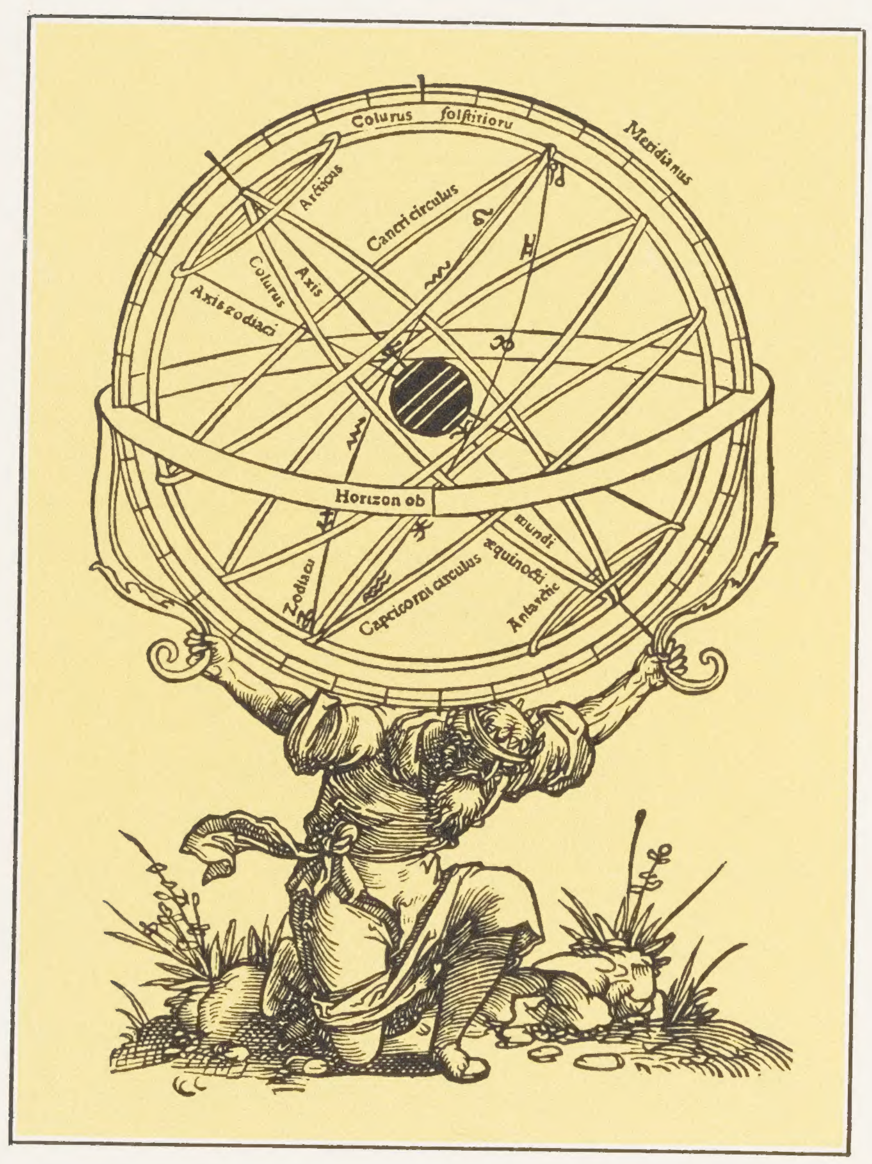

Madeline Copp
and Ronald Brashear

March 6, 2018

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2 vols.

2 vols

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ANALYTICAL INSTITUTIONS,

IN FOUR BOOKS:

ORIGINALLY WRITTEN IN ITALIAN,

BY

DONNA MARIA GAETANA AGNESI,

PROFESSOR OF THE MATHEMATICKS AND PHILOSOPHY IN THE UNIVERSITY OF BOLOGNA.

TRANSLATED INTO ENGLISH

BY THE LATE

REV. JOHN COLSON, M.A. F.R.S.

AND LUCASIAN PROFESSOR OF THE MATHEMATICKS IN THE UNIVERSITY OF CAMBRIDGE.

NOW FIRST PRINTED, FROM THE TRANSLATOR'S MANUSCRIPT,

UNDER THE INSPECTION OF THE

REV. JOHN HELLINS, B.D. F.R.S.

AND VICAR OF POTTER'S-PURY, IN NORTHAMPTONSHIRE.

VOLUME THE FIRST,

CONTAINING THE FIRST BOOK.

To which is prefixed,

AN INTRODUCTION BY THE TRANSLATOR.

LONDON:

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1801.

ADVERTISEMENT

BY

THE EDITOR.

THE *Analytical Institutions* of the very learned Italian Lady, *Maria Gaetana Agnesi*, Professor of the Mathematicks and Philosophy in the University of *Bologna*, which were published in two Volumes, Quarto, in the year 1748, are well known and justly valued on the Continent; and there cannot perhaps be a better recommendation of them in this Island, than that they were translated into English by that eminent judge of Mathematical Learning, the late Reverend *John Colson*, M. A. F. R. S. and Lucasian Professor of the Mathematicks in the University of *Cambridge*. That learned and ingenious man, who had obliged his Country with an English Translation of Sir ISAAC NEWTON's Fluxions, together with a Comment on that profound work, in the year 1736,—and was well acquainted with what appeared on the same subject, in the course of fourteen years afterward, in the writings of those very ingenious men, *Emerson*, *Mac Laurin*, and *Simpson*,—found, after all, the *Analytical Institutions* of *Agnesi* to be so excellent, that he was at the pains of learning the Italian Language, at an advanced age, for the sole purpose of translating that work into English; that the British Youth might have the benefit of it as well as the Youth of Italy.

This

This great deſign he lived to accompliſh; and had actually tranſcribed a fair copy of his Tranſlation for the preſs, and begun to draw up propoſals for printing it by ſubſcription. And, in order to render it more eaſy and uſeful to the Ladies of this Country, (if indeed they can be prevailed upon by his perſuaſion and encouragement, to ſhow to the world, as they eaſily might, that they are not to be excelled by any foreign Ladies whatever, in any valuable accompliſhment,) he had deſigned and begun a popular account of this work, under the title of *The Plan of the Lady's Syſtem of Analyticks;* explaining, article by article, what was contained in it. But this he did not live long enough to finiſh, nor indeed to give more than a rough draught of it ſo far as article 256 of the firſt Book.

In this ſtate the Manuſcript remained many years; and, conſidering the great expenſe which, in the preſent times, attends the printing of ſuch a work, probably might have remained many more, had it not been for the active and liberal ſpirit of Mr. BARON MASERES; who, whether we conſider his own ingenious and extenſive labours in the Mathematicks, or the encouragement which he gives to others who employ their talents in that way, well deſerves what Sir ISAAC NEWTON ſaid of Mr. *Collins*, the great encourager of Mathematical Learning in his time —*Vir in Rem Mathematicam promovendam natus* *. But this commendation is far ſhort of the deſerts of the Patron of this Work. While he ſets a due value upon *Arts* and *Sciences*, he is highly ſenſible of the much greater importance of REVEALED RELIGION, and *well-conſtituted Government*, to the happineſs of mankind; and is no leſs pious and loyal than he is learned and liberal. To the truth of theſe aſſertions every one who is acquainted with him will readily bear teſtimony; and they might be ſupported likewiſe by paſſages from various Books which

* See Comm. Epiſtol. Edit. 1722, p. 148.

are

are well known to be productions of his pen, although ſome of them bear not his name. But I forbear quotations from his works in this place, that I may not, on the one hand, hurt the modeſty of a Friend, nor, on the other, give occaſion to the captious and malevolent to ſay I offer incenſe to my Patron.

When the BARON had reſolved to bear the whole of the expenſe of a handſome Edition of theſe *Inſtitutions*, he was pleaſed to deſire me to ſuperintend the printing of them: to which I readily conſented, in conſequence of favours received from him, and with the hope that I might render ſome little ſervice to the readers of this work, by taking care that it ſhould be correctly printed, which is a matter that requires more time and attention than moſt are aware of, who have not experienced it.

But, beſides correcting the errors of the preſs, it was neceſſary to correct many little ſlips of the pen, and inaccuracies, which I found in the Copy. For, notwithſtanding it was fairly tranſcribed for the preſs in Mr. *Colſon's* own hand-writing, it had evidently been written in haſte, and wanted reviſion; and undoubtedly would have received it from him, if he had lived to ſuperintend the printing of it himſelf. Of theſe inaccuracies, a few were in the language, but more in the mathematical part, where, although I ſeldom found any wrong concluſion, I found many miſtakes in the ſigns and exponents of quantities, as well as omiſſions of numbers and quantities, and ſometimes of whole clauſes. Some of theſe miſtakes I was enabled to correct by means of the foul ſheets on which the Tranſlation was firſt written; but finding errors in them alſo, (ſome of which, I doubt not, were occaſioned by preſs errors in the original, a copy of which I could never obtain),

obtain *,) I ſaw no way of ſatisfying myſelf, but to undertake the labour, great as it was, of examining and recomputing every operation in which I ſuſpected or diſcovered any error: and this was frequently the caſe in the ſecond Volume. In ſhort, my endeavour has been to preſent this Tranſlation to the Public faithfully as the worthy old Profeſſor made it, and would have rendered it, if he had lived to publiſh it; altering nothing in it but the miſtakes before mentioned, nor inſerting any thing of my own but what is included within theſe marks [ ].

With reſpect to the ſtyle of this Tranſlation, ſome of the ſentences, no doubt, might have been better turned; yet the meaning is, in general, plain enough, which is all that is requiſite in books of this kind.

It has been mentioned above, that the Introduction was left unfiniſhed by Profeſſor *Colſon:* I have continued it to the end of the firſt Volume; diſtinguiſhing what I have written from what was found in his Manuſcript by putting it in brackets.

It appears by a paſſage in the Manuſcript of the Introduction, that Mr. *Colſon* intended to make ſome additions to this Work; but what theſe additions were to be is not mentioned. Yet I conjecture that they were to be ſome eaſy pleaſant Queſtions, with their Solutions, in the manner which he has ſhown in Sect. VI. of his Comment on Sir Isaac Newton's Fluxions; merely to exerciſe the learner in the rules given in theſe Inſtitutions, and not to contain any new rules, or additional matter; for he has called this Work of *Agneſi*, *A Complete Syſtem of Analyticks* †. And finding a ſhort Paper of this kind in his handwriting, I have inſerted it at the end of the ſecond Volume.

* In the year 1799, I employed two days in making inquiries amongſt the bookſellers of London, from one end of the city to the other, for a Copy of the Original, without ſucceſs.

† See the Introduction, p. i.

That

That theſe *Inſtitutions*, conſidering the great quantity of valuable matter contained in them, the judicious manner in which it is arranged, and the perſpicuity with which it is explained, will be eſteemed, by all candid judges, as the moſt valuable work of the kind that has appeared in our language, need not be doubted. Inſtances of the ſuperiour ſkill of the Author may be found in various parts of her Work, more eſpecially in the Fourth Book, where it appears in the conſtruction of ſome fluxionary equations without a ſeparation of the variable quantities, —in the ſeparation of the variable quantities in others,---and in the reduction of others in which there are ſecond and third fluxions to equations having firſt fluxions only. A ſingle inſtance of her great ſkill may ſerve to gratify the reader, and, for the ſake of brevity, is all that I ſhall produce in this place. It is taken from the beginning of the fifth Article of the firſt Section of the Fourth Book, where ſhe ſhows that the equation of the fluents of $y^r\dot{y} = x^n\dot{y} + yx^{n-1}\dot{x}$ is $fny^{r+n-1}\dot{y} = x^n y^n \pm b$; which, by only writing x for y and y for x, is the ſolution of the equation $y^n\dot{x} + xy^{n-1}\dot{y} = x^r\dot{x}$; from which the ſolution of the equation $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m\dot{x}}{ay^n}$ is moſt eaſily obtained. This equation is taken from page 289 of the ſecond Volume of *Simpſon's* Fluxions, (publiſhed in the year 1750,) who has there expreſſed his opinion, That the only caſe in which this equation admits of a ſolution "by multiplying, or dividing it, by ſome power or product of the quantities concerned," is, when $n = 1$: whereas *Agneſi* has given a general ſolution by that method *. What is here ſaid is only to

* I am aware that a ſolution of this equation has, of late, been given by ſeveral ingenious perſons of this Country; which, however, ſome of them may ſee reaſon to reviſe.

prove the great ſkill of *Signora Agneſi*, and not with any intent to leſſen the reputation of Mr. *Simpſon;* for whoſe memory and abilities I have the higheſt reſpect, eſteeming him as one of the greateſt Mathematical Geniuses that this Country has produced ſince the time of Sir ISAAC NEWTON.

It may perhaps be objected to theſe *Inſtitutions*, That there are a number of Mechanical and Phyſical Problems to be met with, in ſome Treatiſes of Fluxions in our language, which are not found here. The anſwer is, That ſuch Problems are properly placed in Treatiſes of *Mathematical Philoſophy;* but, as the ſolutions of them require a knowledge of Mechanicks, and Natural Philoſophy, they could not, with any more propriety, be admitted into an Elementary Treatiſe of Fluxions, than the Problems of meaſuring Land, or of taking Heights and Diſtances, could be admitted into *Euclid's* Elements of Geometry.

But here I would not be underſtood to inſinuate that theſe *Inſtitutions* are ſo perfect as to admit of neither improvement nor addition: on the contrary, I have obſerved that ſome of the inveſtigations might be made in a ſimpler manner; and that the *Methods of finding the Roots of numerical Equations by Approximation*, — *Of ſolving literal and fluxionary Equations by infinite Series*,---and *Of comparing together homogeneous Fluents*, are wanting in them; all which might be contained in a few ſheets, and which, if added to this Work, would ſave the learner the expenſe of money and time in procuring and reading a number of books on theſe ſubjects. Theſe Methods therefore, together with Notes on ſeveral parts of the Work, I purpoſe to draw up, under the title of *A Sup-*

A Supplement to Maria Agneſi's Analytical Inſtitutions; to be printed with the ſame type, and on the ſame kind of paper, as this Work; if health and leiſure ſhould permit, and if it ſhould appear to be deſired by Mathematical Readers.

The wonderful ſagacity which appears in theſe *Inſtitutions*, and the ſingular circumſtance that ſo large a work of this kind was performed by a Lady, raiſed in me a wiſh to obtain a particular account of the Author; but the confuſion and miſery which have been brought upon a great part of Europe, and particularly upon Italy, by the French Revolution, have deprived me of the means of getting authentic information reſpecting this *Phænomenon* of Literature from the Univerſity of *Bologna*, of which ſhe was once ſo bright an ornament. All the information I have been able to get of her, (beſides what appears in her excellent Work, and ſome juſt encomiums on her ſkill which I have ſeen in foreign books,) I have inſerted in the following pages; ſuppoſing that the reader would be no leſs deſirous than myſelf of any authentic information reſpecting ſo amiable and ſo extraordinary a perſon. The account comes, indeed, by way of France; yet, as there is no viſible motive for the writers of it to deviate from truth in what they have related of her, I ſee no reaſon for diſbelieving it.

I have alſo inſerted the Teſtimony given by Dr. *Saunderſon* to the great genius and ſkill of Mr. *Colſon*; conceiving that it might prove uſeful information to the junior readers of theſe *Inſtitutions*.

I have only to requeſt of the candid reader that, if, notwithſtanding

the care I have taken in correcting the prefs for this Work, any errors have efcaped me, (and in printing a work of this kind it is hardly poffible but fome will efcape unnoticed,) he will correct them himfelf, and kindly excufe the omiffion.

JOHN HELLINS.

Potter's-Pury,
September 29th, 1801.

SOME

SOME ACCOUNT OF *MARIA AGNESI*,

THE AUTHOR OF THESE *ANALYTICAL INSTITUTIONS*.

IN the Appendix to the XXXIIId Volume of the Monthly Review, pages 516 and 517, is an Account of *Maria Agnesi*, taken from one of M. *De Brosses*' Letters on Italy, which is nearly the same in substance, but not in perspicuity, with what is here printed.

' Letter X.—The account given by Monsieur De Brosses, in the 10th Letter, of a kind of literary phænomenon that he met with in this journey, is so remarkable that we cannot avoid transcribing it. This was a young lady of *Milan*, about eighteen or twenty years of age, named *la Signorina Agnesi*, whom he calls *a walking Polyglott*, and who, not content with knowing all the oriental languages, undertook to maintain a *Thesis* in any of the sciences against any one who should choose to dispute upon it with her. At a *Conversatione* to which our traveller [Monsieur De Brosses] and his nephew were invited, they found about thirty persons, of several different nations of Europe, sitting in a circle, and *la Signorina Agnesi*, with her little sister, seated under a canopy. She could hardly be reckoned handsome; but she had a fine complexion, and an air of great simplicity, softness, and feminine delicacy.'

" I had conceived (says the President *,) when I went to this conversation-party, that it was only to converse with this young lady in the usual way, though on learned subjects; but, instead of this, Count *Belloni* (who had introduced me to it,) made a fine harangue to the lady in *Latin*, with the formality of a college-declamation. She answered with great readiness and ability in the same language; and they then entered into a disputation, still in the same language, on the origin of fountains and on the causes of the ebbing and flowing which is observed in some of them, like the tides in the sea. She spoke like an angel on this subject; and I never heard it treated in a manner that gave me more satisfaction. Count *Belloni* then desired me to enter with her on the discussion of any other subject I should choose to pitch upon, provided that it related to Mathematicks or Natural Philosophy. This proposal alarmed me a good deal,

* M. *De Brosses* was first President of the Parliament of Dijon, and Member of the *Royal Academy of Inscriptions and Belles Lettres* of Paris. According to the Monthly Reviewer, he travelled in Italy about the year 1740: from which it follows that *Agnesi* was about 28 years of age when her *Analytical Institutions* were published.

as

as I found it was expected that I ſhould hold a converſation in the Latin language, with which I had no longer that familiar acquaintance and readineſs at ſpeaking it, which in the days of my youthful ſtudies I had formerly poſſeſſed. However, I made the lady the beſt excuſes I could for my want of ſufficient ſkill in the Latin language to make me worthy of converſing in it with her, and hoped ſhe would over-look the incorrect expreſſions I might happen to make uſe of in the courſe of the diſcuſſion ; and we then entered, firſt, into an inquiry concerning the manner in which the ſoul receives impreſſions from corporeal objects, and in which thoſe impreſſions are communicated from the eyes, and ears, and other parts of the body on which they are firſt made, to the organs of the brain, which is the general *ſenſorium*, or place in which the ſoul receives them ; and we afterwards diſputed on the propagation of light and the priſmatick colours. *Loppin* then diſcourſed with her on *tranſparent bodies*, and on *curvilinear figures* in Geometry, of which laſt ſubject I did not underſtand a word. Loppin ſpoke in French ; and the lady begged to be permitted to anſwer him in Latin, fearing that ſhe ſhould not be able to recollect the proper French technical names of the ſeveral ſubjects which they ſhould have occaſion to conſider.

" She ſpoke wonderfully well on all theſe ſubjects, though ſhe could not have been prepared before-hand to ſpeak upon them, any more than we were. She is much attached to the Philoſophy of Sir ISAAC NEWTON : and it is marvellous to ſee a perſon of her age ſo converſant with ſuch abſtruſe ſubjects. Yet, however much I may have been ſurprized at the extent and depth of her knowledge, I have been much more amazed to hear her ſpeak Latin (a language which ſhe certainly could not often have occaſion to make uſe of,) with ſuch purity, eaſe, and accuracy that I do not recollect to have ever read any book in modern Latin that was written in ſo claſſical a ſtyle as that in which ſhe pronounced theſe diſcourſes. After ſhe had replied to *Loppin*, the converſation became general, every one ſpeaking to her in the language of his own country, and ſhe anſwering him in the ſame language : for her knowledge of languages is prodigious. She then told me that ſhe was ſorry that the converſation at this viſit had taken that formal turn of an *Academical Diſputation*, declaring that ſhe very much diſliked ſpeaking on ſuch ſubjects in numerous companies ; where, for one perſon who received amuſement from the diſcuſſion of them, there were often twenty who were tired to death by it ; and that therefore ſuch ſubjects were only fit to be entered-upon in ſmall companies of two or three perſons, who had all the

the ſame taſte for diſcuſſing them. This obſervation, I thought, was very juſt, and was a proof of the ſame good ſenſe and diſcernment which had appeared in her former learned diſcourſes. I was ſorry to hear that ſhe was determined to go into a Convent, and take the veil: which was not from want of fortune, (for ſhe is rich,) but from a religious and devout turn of mind, which diſpoſes her to ſhun the pleaſures and vanities of the world. After the converſation was finiſhed, her little ſiſter played on the harpſichord, with the ſkill of a Rameau, firſt, ſome of Rameau's pieces of muſic, and then ſome pieces of her own compoſition, and concluded by ſinging ſome airs and accompanying her voice on the inſtrument."

M. *Montucla* ſpeaks of *Maria Agneſi*, and of her *Analytical Inſtitutions*, to the following effect, in his *Hiſtoire des Mathématiques*, Volume II, page 171.

" Beſides the foregoing Authors I ought to mention on this occaſion, with much commendation, the *Analytical Inſtitutions* of a learned Italian lady of the name of *Maria Gaetana Agneſi*, which is a work of ſuch merit that ſome female mathematician of France (for we alſo have ſome ladies of that deſcription among us,) would have done well to give us a French tranſlation of it. We cannot behold without the greateſt aſtoniſhment a perſon of a ſex that ſeems ſo little fitted to tread the thorny paths of theſe abſtract ſciences, penetrate ſo deeply as ſhe has done into all the branches of Algebra, both the common and the tranſcendental, or infiniteſimal. She has ſince retired to a cloiſter: and, though we do not preſume to cenſure her conduct in this ſtep, (which we muſt ſuppoſe to proceed from the pureſt and ſincereſt piety,) we cannot but lament that ſhe ſhould have thus deprived the learned world of the uſeful improvements in Literature which her genius and knowledge would have enabled her to communicate to it, not only on ſubjects of a mathematical nature, but on many others of a different kind, in which ſhe had become eminent."

In the Index to the Volume above mentioned, M. *Montucla*, at the name *Agneſi*, refers alſo to the third Volume of his work, which is not yet publiſhed.

Maria Agneſi and her *Analytical Inſtitutions* are mentioned alſo in a note in page 179 of a work intitled " *An Eſſay on the Learning, Genius, and Abilities of the Fair-Sex: proving them Not Inferior to Man, from a Variety of Examples, extracted from Antient and Modern Hiſtory. Tranſ-*

lated from the Spanish of ' *El Theatro Critico.*' London 1774." What is there said of her is to the effect following:

" A learned Italian lady of our own times is *Signora Agnesi* *, daughter of a creditable tradesman in Milan, famed throughout all Europe for her knowledge of the learned languages and for being the author of a profound treatise of Algebra, intitled *Analytical Institutions,* which, besides many eulogiums bestowed on her by several Scientifical Societies, has gained her a Professorship of Mathematicks in the University of Bologna. Neither her inclination to these favourite intellectual pursuits, nor a desire of preserving and increasing the fame she had acquired by her attainments in them, nor the intreaties of her father have been able to stifle the call from heaven which she conceives herself to have felt in her child-hood to dedicate herself to a monastick life amongst the nuns known by the name of *The Blue Nuns,* than which there are few orders in the Church of Rome subject to rules of greater severity. Since her father's death she has given herself up to the most sublime devotion, and has sacrificed to christian self-denial all those enjoyments in the society of the world to which her fine qualities and literary attainments had already introduced her amongst the most respectable part of mankind."

DR. SAUNDERSON'S TESTIMONY OF THE GENIUS OF MR. *COLSON.*

DR. NICHOLAS SAUNDERSON, *Lucasian* Professor of the Mathematicks in the University of Cambridge, and Fellow of the Royal Society, speaking of Mr. *Colson* in his *Algebra,* Vol. II. p. 720, has these words:

—" The learned Mr. *John Colson,* a gentleman whose great genius and known abilities in these sciences I shall always have in the highest admiration and esteem."

Mr. *De Moivre* also has, on several occasions, spoken of the great skill of Mr. *Colson;* but, for want of books, I cannot quote his words. However, Dr. *Saunderson's* Testimony, and the office which Mr. *Colson* afterward held in the University of *Cambridge,* are sufficient vouchers of his ability.

* In the Note above referred to, which seems to be a bad translation of a passage in a book intitled ' *Observations sur l'Italie,* &c,' her name is erroneously printed *Anglese.* I have therefore given the same Account in better English, as it was communicated to me by Mr. *Baron Maseres;* to whom also I am obliged for all the rest that is here printed concerning this very extraordinary person. J. H.

THE

THE AUTHOR'S DEDICATION.

TO

HER SACRED IMPERIAL MAJESTY,

MARIA TERESA OF AUSTRIA,

EMPRESS OF GERMANY, QUEEN OF HUNGARY, BOHEMIA, &c. &c.

AMONG the various arguments I revolved in my mind, inducing me to hope, that Your Sacred Majeſty, according to your great condeſcenſion, would vouchſafe to receive favourably this Work of mine, which is proud to ſhelter itſelf under your auguſt name, and humbly to crave your gracious patronage and protection; among all theſe arguments, I ſay, none has encouraged me ſo much as the conſideration of your ſex, to which Your Majeſty is ſo great an ornament, and which, by good fortune, happens to be mine alſo. It is this conſideration chiefly that has ſupported me in all my labours, and made me inſenſible to the dangers that attended ſo hardy an enterpriſe. For, if at any time there can be an excuſe for the raſhneſs of a Woman, who ventures

to aſpire to the ſublimities of a ſcience, which knows no bounds, not even thoſe of infinity itſelf, it certainly ſhould be at this glorious period, in which a Woman reigns, and reigns with univerſal applauſe and admiration. Indeed, I am fully convinced, that in this age, an age which, from your reign, will be diſtinguiſhed to lateſt poſterity, every Woman ought to exert herſelf, and endeavour to promote the glory of her ſex, and to contribute her utmoſt to increaſe that luſtre, which it happily receives from Your Majeſty; who, having diffuſed, on all ſides, the fame and admiration of your actions, have obliged Mankind to apply to you, with much greater reaſon, what has been ſaid of ſome of the antient Cæſars;—that, by the juſtice and clemency of your Government, you are an honour to human nature, and a near reſemblance of the divine. To thoſe who, zealous for the glory of our ſex, ſhall faithfully tranſmit to poſterity the memory of your deeds; to thoſe (I ſay) I muſt leave to commemorate, how each accompliſhment of the mind is united in Your Majeſty with the moſt engaging gracefulneſs of perſon; to thoſe I ſhall leave the arduous taſk to deſcribe, the ſtrength of your underſtanding, the extenſiveneſs of your genius, but, above all, that ſignal fortitude, that invincible courage and conſtancy of mind, by which you derived freſh vigour, as it were, from your perils and perſecutions themſelves; and, after having been ſo ſeverely tried by the hand of Providence at the beginning of your reign, gave at laſt ſo happy a reverſe to your affairs. Neither will they fail to celebrate the engaging ſweetneſs of your temper, your humane and compaſſionate diſpoſition, nor that generous condeſcenſion with which, amidſt the hurry and tumult of

arms, you cherifh and protect the arts and fciences; being duly fenfible how greatly thefe redound to the public welfare; and that by thefe the minds of men are forcibly excited to the purfuit and practice of every focial virtue. Hence it was, that the Sciences fo early took poffeffion of your mind, and that you became well acquainted with the whole circle of them. And though the bufy cares and interruptions of Empire may have withdrawn you from your more ftudious applications, (Heaven having thought it too fmall a commendation for you, to be called the moft knowing and learned Woman of your age,) yet ftill your love of truth is not the lefs fervent; fo that whoever employ themfelves in the fearch of it, are fure to meet with diftinguifhing marks of your approbation.

Vouchfafe, therefore, Madam, to caft a favourable eye on this Performance of mine, not only as a Work which comprehends the higheft attempts of the human underftanding, but alfo as the greateft tribute it was in my power to offer, to the glory of your aufpicious reign; a reign which feems to revive the memory of former heroines, only to render your magnanimity, prudence, and good fortune, the more eminently confpicuous by the comparifon. And if the Volume of Mufic, which my Sifter has had the honour of prefenting to Your Majefty, has been fo fortunate as to excite your voice to melodious accents; let this be fo happy as to have the defired effect, of employing fometimes the fagacity and penetration of your underftanding. As nothing more remains, but to implore of Heaven a long and happy continuance of your glorious reign, for the felicity of the many nations fubject to your command; I

therefore proſtrate myſelf, with all humility, at the foot of your Throne, and am

Your Majeſty's

moſt humble,

moſt obedient,

and faithful ſervant,

MARIA GAETANA AGNESI.

THE

THE AUTHOR'S PREFACE

TO

THE READER.

THERE are few ſo unacquainted with Mathematical Learning, but are ſenſible the Study of Analyticks is very neceſſary, eſpecially in our days; they cannot but be apprized what improvements have already been made by it's means, what are ſtill making every day, and what may be yet expected in time to come. For which reaſon I ſhall not amuſe myſelf with making unneceſſary encomiums on this ſcience, which ſtands in no need of any ſuch recommendations, and much leſs of mine. But, notwithſtanding the neceſſity of this ſcience appears ſo evident as to excite our youth to the earneſt ſtudy of it; yet great are the difficulties to be overcome in the attainment of it. For it is very well known, that perſons able and willing to teach it are not to be found in every city, at leaſt not in our Italy; and every one that would be glad to learn has not the means of travelling into diſtant countries, in queſt of proper maſters. This I know by my own experience, as I muſt ingenuouſly confeſs; for, notwithſtanding the ſtrong inclination I had to this ſcience, and the great application I made uſe of to acquire it; I might ſtill have been loſt in a maze of inextricable difficulties, had I not been aſſiſted by the ſecure guidance and ſage direction of the very learned Father *Don Ramiro Rampinelli*, Monk of the Olivetan Order, and now Profeſſor of

 the

the Mathematicks in the Royal Univerſity of *Pavia*; to whom I acknowledge myſelf indebted for what little progreſs I may poſſibly have made in this kind of ſtudy; on whoſe deſerved praiſe I ſhall forbear to inſiſt, it being unneceſſary to a perſon of his fame and merit, and offenſive to his known, but perhaps too rigid, modeſty. True it is, the aforeſaid inconvenience may, in ſome meaſure, be removed, by having recourſe to good books, written with perſpicuity, and (what is above all) in a proper method. But though what relates to the ſubject of Analyticks may have already been treated of, and is to be found in print; yet as theſe pieces are ſcattered and diſperſed in the works of various authors, and particularly in the *Leipſic* Acts, the Memoirs of the Royal Academy of Sciences at *Paris*, and in other foreign Journals; ſo that it is impoſſible for a beginner to methodize the ſeveral parts, even though he were furniſhed with all the books neceſſary for his purpoſe: this conſideration induced the celebrated Father *Renau* to publiſh that moſt uſeful Work, intitled *L'Analyſe demontrée*, a work deſerving the higheſt commendation. After which, I am very ſenſible, that theſe Inſtitutions of mine may ſeem, at firſt ſight, to be needleſs, ſo many learned Men having thus amply provided for the occaſions of the Public. But, as to this point, I deſire the candid reader to conſider, that, as the Sciences are daily improving, and, ſince the publication of the aforementioned book, many important and uſeful diſcoveries have been made by many ingenious writers; as had happened likewiſe to thoſe who had written before them: Therefore, to ſave ſtudents the trouble of ſeeking for theſe improvements, and newly-invented methods, in their ſeveral authors, I was perſuaded that a new Digeſt of Analytical Principles might be uſeful and acceptable. The late diſcoveries have obliged me to follow a new arrangement of the ſeveral parts; and whoever has attempted any thing of this kind muſt be convinced, how difficult it is to hit upon ſuch a method as ſhall have a ſufficient degree of perſpicuity, and ſimplicity, omitting every thing ſuperfluous, and yet retaining all that is uſeful and neceſſary; ſuch, in ſhort, as ſhall proceed in that natural order, in which

conſiſts

consists the closest connexion, the strongest conviction, and the easiest instruction. This natural order I have always had in view; but whether I have always been so happy as to attain it, must be left to the judgment of others.

In the management of various methods, I think I may venture to say, that I have made some improvements in several of them, which I believe will not be quite devoid of novelty and invention. To these the judicious Reader may give what weight he pleases. It was never my design to court applause, being satisfied with having indulged myself in a real and innocent pleasure; and, at the same time, with having endeavoured to be useful to the Public.

In the Second Volume, in which I treat of the Integral Calculus, or what is also called the Inverse Method of Fluxions, the Reader will meet with a speculation entirely new *, and no where before published, concerning *Multinomials*. For this I am indebted to the celebrated Count *James Riccati*, a gentleman who has greatly deserved of every branch of literature, and whose merit is well known to the learned world. He was pleased to communicate this to me, which I take as a favour beyond my deserts; and for which both the Public and myself are bound to give him our thanks.

To conclude: As it was not my intention, at first, that the following Work should ever appear in public; a work begun and continued in the *Italian* tongue, purely for my own private amusement, or, at most, for the instruction of one of my younger Brothers, who possibly might have a taste for mathematical studies; and as I had not determined to send it abroad till after it was pretty far advanced, and had grown to the size

* It does not appear to me, that any thing can be done by this new method, which may not be done as well, or better, without it. J. H.

of

of a juſt volume; then I thought I might be excuſed the trouble of tranſlating it into Latin, (a language which ſome may imagine is more ſuitable to works of this nature,) eſpecially as I had the example of ſo many famous Mathematicians, as well Italians as others, who have publiſhed their Mathematical Works in their own mother-tongues. Nor could I eaſily overcome my natural indolence, in ſubmitting to the drudgery of tranſlating that into Latin which I had already compoſed in *Italian*. Far am I therefore from laying the leaſt claim to any merit ariſing from that purity and elegance of ſtyle, which in ſubjects of a different nature may be laudably attempted; being fully ſatisfied if I have always expreſſed myſelf, as I ſincerely endeavoured, in a plain, but clear and intelligible manner.

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INTRODUCTION.

THAT we ſhould receive from Italy, the Mother of Arts, a complete Syſtem of Analyticks, is not ſo much to be wondered at; knowing we have often had from that quarter very excellent productions in the ſublimer Mathematicks. But, that we ſhould receive ſuch a preſent from the hands of a Lady; from that ſex which, however capable, yet hardly ever amuſe themſelves with theſe ſeverer ſtudies; is, indeed, very wonderful and ſurpriſing. Yet ſo it is in fact: a very learned, ingenious, and celebrated Lady of *Milan*, by name *Donna Maria Gaetana Agneſi*, a member of the Univerſity of *Bolonia*, and lately advanced by the Pope to a Profeſſorſhip in Mathematicks and Philoſophy in the ſame Univerſity, has publiſhed a Treatiſe in Italian, in two volumes *quarto*, which ſhe calls *Analytical Inſtitutions for the Uſe of the Youth of Italy*; of which ſhe was pleaſed to preſent a Copy to the Royal Society of *London*. This Copy I had the curioſity to inſpect, and thought it might be a proper way of returning the Author's compliment, to have an Account of the work drawn up and read to the Society, and perhaps printed in the Philoſophical Tranſactions, as has often been the practice on ſuch occaſions. This Account, therefore, I undertook to draw up, having the conſent and approbation of our worthy Preſident. But when I came to look into the work more cloſely, I ſoon enlarged my ſcheme;

and, inſtead of barely taking the Plan, or giving an Account of it, I thought it highly deſerved to be tranſlated into our own language, that the Youth of *England* might likewiſe enjoy the benefit of it. This determined me then to attempt it's tranſlation, though I well knew how unequal I was to the taſk. I confeſs I alſo entertained ſome diſtant hopes, that it might excite the curioſity of ſome of our *Engliſh* Ladies; that it might raiſe an emulation in them, a laudable ambition to promote the glory of their country, with a generous reſolution not to be outdone by any foreign ladies whatever. They want no genius or capacity for the ſciences, and have undoubtedly as good abilities as the Ladies of *Italy*. They ſeem only to want to be properly introduced into theſe ſtudies, to be convinced of their uſefulneſs and agreeableneſs, and to prevail on themſelves to uſe the neceſſary application and perſeverance. They have here a noble inſtance before them, of what the ſex is capable to perform, when their faculties are exerted the right way. And they may be fully perſuaded, that what one lady is able to write, other ladies are able to imitate, or, at leaſt, to read and underſtand. With not much more pains and induſtry than what they muſt be at, to be expert at Whiſt or Quadrille, they may become miſtreſſes of this ſcience; which they will find to be much more innocent, more diverting and agreeable, and to have infinitely more amuſing variety than thoſe, or any other games whatever. Indeed, this is rather to be eſteemed a game, or a diverſion, than a ſtudy; but then it is a game of ſkill, without any mixture of chance, like Cheſs and ſome other ingenious games: and parties of two, or more, may play at it together, by propoſing curious queſtions to one another alternately, to their great diverſion and improvement. The games of Whiſt, Quadrille, Back-gammon, &c. and all other games in which chance predominates, but ſkill is alſo required to convert the events of chance to the beſt advantage; theſe are only particular caſes of this general game or art, and ought always to be regulated by it. For, in all inſtances, Analyticks may be uſed to diſcover the odds, or degrees of probability, which are for, or againſt, the happening of any particular event, and ſo the chance may be made equal on all ſides, notwithſtanding a ſuperiority of ſkill on one ſide. And thus all games of chance may be made fair and equal; and the well-meaning gameſter will not be impoſed on by ſharpers, who, by much obſervation, rather than by ſkill in Analyticks, always know what they call the beſt of the lay, or always have the odds on their ſide.

But this is the leaſt recommendation of this ſcience. The improvement of their minds and underſtandings, which will neceſſarily ariſe from hence, is of much greater importance. They will be inured to think clearly, cloſely, and juſtly; to reaſon and argue conſequentially, to inveſtigate and purſue truths which are certain and demonſtrative, and to ſtrengthen and improve their rational faculties. Now that theſe, and all other readers, may attain theſe advantages with as little trouble as poſſible, I ſhall endeavour to draw out the Plan of this Work at full length, and in a popular manner, inſerting ſome uſeful Obſervations to explain the Art itſelf; ſo that the Work, when publiſhed, may be eaſily read and apprehended, by ſuch as will peruſe it with the neceſſary diligence and attention.

The ſubject of the Work is Analyticks, or the general Science of Computation or Calculation. That is, the Art of reſolving all kinds of Mathematical Queſtions, by finding or computing unknown numbers, or quantities, by the means of others that are known or given. Theſe computations are performed either by common numbers, and then the ſcience is called *Arithmetick:* or by general numbers or arbitrary ſymbols of quantities, which are commonly the letters of the alphabet, and then it is uſually called *Algebra:* or by lines and geometrical figures, which are likewiſe the ſymbols of quantities, and then it is called *Geometry:* or, laſtly, by all theſe conjunctly and indifferently, and then it will properly be called *Analyticks.* All theſe ſciences our Author teaches and explains promiſcuouſly, but in good order and method, at leaſt the higher and more difficult parts of them; for ſhe requires, as very reaſonably ſhe may, that the learner ſhould come prepared with a pretty good ſtock of common Arithmetick, with a competent knowledge of the firſt elements of Geometry, and with ſome inſight into the ſimpler properties of the Conic Sections. Theſe are acquiſitions with which they may be eaſily furniſhed out of the common mathematical books on theſe ſubjects; which will then prepare the way for an eaſy acceſs to her ſublimer ſpeculations. Now, to enter upon our intended Plan. The Author divides her ſubject into two Tomes, or Volumes; in the firſt of which ſhe treats of the common, ordinary, and finite quantities, and their repreſentatives, whether numbers, general ſymbols, or lines. In the ſecond Volume ſhe explains the nature of what ſhe calls *Infiniteſimals*, or infinitely ſmall Quantities; proves their comparative exiſtence, and ſhows their uſe and

application. This is the grand divifion of the whole Work, which is again divided into four *Books*, and every Book is fubdivided into it's number of *Sections*, according to the nature of the feveral fubjects they treat of. Laftly, there is a further fubdivifion of the Sections into *Articles*, which are numbered without interruption from the beginning to the end of each Book, and which we fhall alfo obferve and enumerate in our explications of them.

PLAN.

THE firft Section of the firft Book is concerning the primary Notions and Operations of the Analyfis of finite Quantities; in which are contained the following Articles. After a fhort Preface concerning the nature of Analyfis, the Author obferves,

1. That it's operations are the fame as thofe of common Arithmetick; this operating with numbers, and that with fpecies, that is, with fymbolical numbers or quantities. By which means Algebra has great advantages over Arithmetick; for, in this, the fteps of the operations will be confounded and loft by the fubfequent ones, but in Algebra they may be preferved, as they are often not actually performed, but only infinuated by proper fymbols; it is alfo more univerfal, and works indifferently with known or unknown quantities.

2. Here the diftinction of pofitive and negative numbers, or quantities, is explained. Negative quantities are not in nature, but depend only on the manner of conceiving them. They are merely artificial, and introduced to fave needlefs repetitions and diftinctions, by which we can confider the oppofite operations of Addition and Subtraction under one general view and comprehenfive idea. In Geometry, they are reprefented by lines drawn oppofite ways. If pofitive lines proceed to the right-hand, then negative ones will be to the left, with the fame direction; or if pofitive ones are upwards, then negative will be downwards.

Then

3. Then different affections of quantities are diftinguifhed, or denoted, by the figns + or —, *plus* or *minus*, placed before them; whether the quantities are reprefented arithmetically, or by common numbers; or elfe algebraically, by reprefentative numbers, that is, by the letters of the alphabet: *plus* being the mark of Addition, and *minus* of Subtraction. And the fign $\pm$ and $\mp$ are ambiguous, but contrary to each other. The equality of quantities is denoted by the mark =, and majority or minority by the marks > or <. Proportion, or equality of ratios, by ::, and infinitely great by ∞.

4. Quantities are *fimple* that are not connected by the figns + or —, and *compound* when they are: of which examples are propofed by the Author.

5. Then is taught the addition of fimple quantities being integers, and explained by a fufficient number of examples: alfo, the ufe of numeral co-efficients is fhown.

6. Likewife, the fubtraction of fimple integral quantities is taught, in which it is fhown that the fign of the quantity to be fubtracted muft always be changed, and the reafon of it, together with examples.

7. Next the Author proceeds to the multiplication of fimple quantities, being integers, whether they are pofitive or negative. Then the product will be reprefented by the connection of the feveral factors, and their co-efficients without any fign between them. And if the factors are pofitive and negative promifcuoufly, like figns will always produce +, and unlike figns —. This fhe demonftrates from the nature of proportion.

8. And whereas raifing of *powers* is a cafe of multiplication; fhe fhows how fimple powers are formed, and conveniently expreffed by their *indices*, or *exponents*, annexed to the roots.

9. Thefe powers are diftributed into *fquares*, *cubes*, *biquadrates*, &c.; that is, into fecond, third, fourth, &c. powers, of which the given number, or *root*, is always the firft power; and they are marked by the exponents 1, 2, 3, 4, &c. refpectively. Their figns are always known by the general rule aforegoing.

10. Then

10. Then comes divifion of fimple quantities, being integers, which is juft the reverfe of multiplication, and refolves, or decompounds, that which the other had compounded; as by the examples.

11. When common letters or quantities are rejected, and the divifion can proceed no further, it muft be infinuated, by making a fraction of what fhall remain.

12. When the figns of the dividend and divifor are the fame, the fign of the quotient muft be pofitive; but when thofe figns are different, the fign of the quotient muft be negative. This proved from the nature of proportion.

13. Whence, in fractions, it is indifferent how the figns are changed in the numerator and denominator, provided the fign of each is changed into it's contrary.

14. The roots of fimple quantities will be extracted, by dividing their exponents by the number which denominates the root to be extracted. As, by 2 for the fquare-root, by 3 for the cube-root, and fo on.

15. If any even root is to be extracted, the fign of that root will be ambiguous; but if an odd root is to be extracted, the fign of that root will be the fame as of the given power.

16. When roots are furd, and cannot be extracted, they are to be infinuated by radical figns or characters.

17. From thefe operations belonging to fimple quantities, the Author proceeds to thofe of compound quantities, or fuch as confift of feveral fimple quantities, connected by the figns + and —. Thus, Addition will be performed by fetting down all the given quantities together promifcuoufly, and then abbreviating the fum as much as may be, and expunging equivalents with contrary figns.

18. In Subtraction, all the figns are changed of the quantity to be fubtracted, and the remainder, or difference, fo found is to be abbreviated as much as may be done.

19. Mul-

19. Multiplication of compound quantities, being integers, depends on the multiplication of fimple quantities; and the procefs is much like the fame operation in common Arithmetick, as the examples fhow.

20. But it is often convenient only to infinuate this multiplication, without actually performing it. And that is done by drawing a line, or *vinculum*, over the feveral factors, and connecting them by putting the mark $\times$, fignifying *Multiplied by*, between them.

21. The powers of compound quantities, as well as of fimple, need not always be actually formed, but may often be conveniently infinuated, by a *vinculum* placed over the root, and a proper index annexed to it. How thefe powers may be actually formed, when occafion requires, is here fhown.

22. The Author prefents us with a general Canon, (being Sir *Ifaac Newton's* Binomial Theorem,) for raifing any binomial quantity, or even multinomial, to any power required; which fhe exemplifies by a fufficient number of examples.

23. The Author proceeds to divifion of compound quantities, being integers, of which fhe makes three cafes. The firft is, when the divifor is fimple and the dividend compound, and the fecond is on the contrary. Thefe are eafily reduced to the foregoing rules.

24. The third cafe is, when both the dividend and divifor are multinomials, and therefore requires a more prolix procefs. In order to which, the terms of each are to be difpofed according to the dimenfions (or powers) of fome particular letter contained in them; that is, they are to form numbers belonging to a fcale, of which that letter is the root, juft as we do in our common Arithmetick, the root of which is ten, and the numbers are difpofed according to the dimenfions of that root. Then the procefs of divifion muft be performed much after the manner of the like procefs in numbers, and which is fufficiently explained by the examples produced. When the dividend cannot be intirely exhaufted, the quotient muft be completed by adding a fraction to it, as in common Arithmetick.

25. The Author proceeds to the extraction of the roots of compound quantities, being integers, and firft of the fquare-root. The terms of the given

quantity

quantity are to be difpofed, as before, in Divifion; and the procefs of extraction will be nearly as the fame operation in numbers. Indeed, her procefs is fomething different in form from the common one, but is very intelligible, and comes to the fame thing. Her examples make it very clear. When the root is furd, and therefore cannot be extracted, it muft be infinuated by a quadratick vinculum.

26. The procefs of the extraction of the cube-root is much after the fame manner, only more operofe, as being a more complicate operation. The examples render it as plain as the nature of the thing will admit.

27. The biquadratick, or fourth root, is extracted in the fame manner.

28. The fifth root, and all higher roots, may be extracted, by forming rules for them, which are found by raifing a binomial to the fame power. For the like was done in forming rules, by which the fquare and cube-roots have been extracted.

29. The Author then proceeds to the algorithm of fractions fimple and compound; obferving that any quantity may be converted into a fraction with a denominator given, if it be multiplied into that denominator: of which fhe produces feveral inftances. For this fee the Examples.

30. Then comes the reduction of fractions to more fimple expreffions, when that can be done, which it is not always eafy to perceive. When the numerator and denominator are each multiplied by the fame quantity, whether fimple or compound, they may each be divided by it again, and a new fraction will arife equivalent to the former. And fo *toties quoties*. This will be a very ufeful reduction; for, in all our calculations, we fhould always ftudy to abbreviate as much as poffible. See the Examples. How thefe common divifors may be found we fhall be taught afterwards.

31. Then is taught reduction of fractions to a common denominator, which in two fractions is performed by the crofs multiplication of each numerator into the denominator of the other, as by the examples. And fo two by two, if there are more, till all are reduced.

32. This

32. This prepares the way for the addition and ſubtraction of fractions; for, if they have not a common denominator, thoſe operations can only be inſinuated, by writing them after one another with their proper ſigns. But, when reduced to a common denominator, their numerators may then be added or ſubtracted, to compleat theſe operations; as by the examples.

33. The multiplication of fractions requires no ſuch preparation, but is performed directly, by multiplying the numerators together for a new numerator, and the denominators together for a new denominator. The product, or fraction thence ariſing, may often be reduced by ſome of the foregoing methods.

34. Diviſion of fractions is reduced to multiplication, by multiplying the dividend by the reciprocal of the diviſor; which reciprocal is, when the numerator and the denominator change places. The quotient thus found will often have occaſion for ſome reduction, as by the examples may be ſeen.

35. As for the extraction of the roots of fractions, whether it be the ſquare-root, the cube-root, &c. the ſaid roots muſt be extracted ſeverally out of the numerator and denominator, and the fraction thence ariſing will be the root of the fraction given. But when ſuch root cannot be extracted, it muſt be inſinuated by placing a radical vinculum before the given fraction, as by the examples.

36. To conclude the Doctrine of Fractions, the Author proceeds to a very curious and uſeful operation, which is, to find the greateſt common diviſor of two quantities or formulas given. Where it may be obſerved, that a formula is a combination of quantities, which may ſerve as a paradigm, or pattern, for all combinations of the like kind. Then, by a proceſs not unlike that in Arithmetick, which is, by ſubtracting one from the other continually and interchangeably as often as can be done, the laſt quantity ſo found will be the greateſt common diviſor of the two given quantities. Now, if thoſe two quantities form a fraction, and the numerator and denominator are each divided by the greateſt common diviſor ſo found, a fraction will thence ariſe equal to the other, but reduced to the ſmalleſt terms. Of this reduction ſhe gives us the proceſs at large, in three ſeveral inſtances.

37. The Author goes on then to the Doctrine of Surds or Radicals, which are ſuch quantities whoſe roots cannot be extracted, yet may often admit of a partial extraction, or may be reduced to ſimpler expreſſions; as by the examples may appear.

38. The reduction of different radicals to radicals with the ſame index, will be performed by finding the leaſt number for a common exponent, by which the given exponents may be divided. Then each radical muſt be raiſed, if neceſſary, till it arrives at that exponent. The examples make it plain.

39. Addition and ſubtraction of radicals is eaſily performed, by writing them one after another with their proper ſigns, and then abbreviating when it may be done.

40. Radical quantities are multiplied by thoſe that are rational, by prefixing the rational to the radical, with ſuch ſign as the Rule of Multiplication requires. And when they are complicate, their product will be found by the ſame rule.

41. Radicals of the ſame denomination, or reduced to ſuch, are multiplied by putting their product under the ſame radical vinculum.

42. If the radicals are affected by rational co-efficients, their product muſt be put before the radical ſo found.

43. When like quadratick radicals are multiplied into each other, the radical ſign will be taken away, and the product will often become rational. Several examples of this are exhibited.

44. A rational co-efficient to a radical may at any time be made to paſs under the radical vinculum.

45. The multiplication of radicals of different kinds may be inſinuated, or they may be reduced to the ſame kind.

46. Diviſion of radicals of the ſame kind is performed by leaving out the radical quantity, and dividing the co-efficients only.

47. If the radicals are of the ſame kind, but not of the ſame quantity, the quantities under the vinculum may be divided, and the quotient put under the ſame vinculum.

48. But

48. But if the radicals are different, they may be reduced to the ſame exponent, and then divided as before. And thus complicate quantities may be divided as in common Diviſion.

49. Then the Author gives us a Rule for extracting the ſquare-root of quantities any how compounded of rational and irrational quantities, and thoſe either numeral or algebräical; which ſhe applies to ſeveral examples.

50. In order to the calculation of powers, which are expreſſed by integer exponents; from any root ſhe forms a geometrical progreſſion of it's powers, beginning from unity, and aſcending one way by poſitive exponents, and deſcending the other way by negative exponents, to ſhow the correſpondence there is between the increaſing powers and their affirmative exponents, and the decreaſing powers and their negative exponents. Then obſerves, that when any power is in the denominator of a fraction, it may be made to paſs into the numerator, and *vice verſâ*, by only changing the ſign of the index.

51. Then, as fractional powers, or roots, are certain intermediate terms, between the integral powers in the foregoing geometrical progreſſion; ſo their exponents muſt be correſponding intermediate terms in the arithmetical progreſſion. And this will obtain in the deſcending progreſſion as well as in the aſcending, and whether the terms are ſimple or compound.

52. Hence the multiplication or diviſion of powers will eaſily be performed by their exponents. For, to multiply them, we muſt add their exponents; and to divide them, we ſubtract the exponent of the diviſor from that of the dividend. This ſhe proves from the nature of proportion.

53. Hence the raiſing of powers, or extracting the roots of any powers, will eaſily be performed by their exponents. For the index of any power muſt be multiplied by the index of the power to which it is to be raiſed; and the index of the given power is to be divided by the index of the root to be extracted.

54. And this obtains as well in compound quantities as in ſimple. For all which reductions ſee the Examples.

55. Another uſeful operation follows, which is that of finding all the linear or ſimple diviſors of any given number or formula; or to reſolve a compound

quantity into the ſeveral quantities of which it is, or may be, compounded by multiplication. The proceſs is exemplified and illuſtrated both in numbers and ſpecies. Indeed, if this could always be done in numbers, it would amount to a very valuable diſcovery, or deſideratum in Analyticks, which is, a method of reſolving a given compound number into the prime numbers of which it is compounded; but though it is only a tentative method, yet, however, it is very uſeful.

56. This is extended to any compound formula, or to a number expreſſed by an indefinite root in an arithmetical ſcale, which may have been formed by the multiplication of ſeveral binomial factors. By this method ſuch a number may again be reſolved into it's factors, by the help of the foregoing operation. And if the number of trials to be made ſhould happen to be too great, the Author ſhows a method of reducing them to a ſmaller number, which is, by changing the root, and ſo exhibiting the given formula by another ſcale.

57. Now, if the firſt term of the given formula ſhould happen to have a numeral co-efficient, it may be convenient (by ſubſtitution) to change it into another formula, or to expreſs it by an equivalent root of another ſcale, the co-efficient of the firſt term of which ſhall be unity.

BOOK I. SECT. II.

Of Equations, and of Plane Determinate Problems.

58. Having explained the firſt principles or operations of Analyticks in the foregoing Section, our Author proceeds to the grand inſtrument of the art of computation, which is equation. This is either when ſome of the terms placed before the mark of equality, are collectively equal to all the terms on the other ſide, called the *homogeneum comparationis*; or when the whole are one ſide, and equal to nothing on the other ſide; inſinuating that the affirmative and negative are equal, and ſo deſtroy one another. She explains likewiſe what is meant by the law of *homogeneity*.

59. She

59. She tells us what a *Problem* is, and what is the diſtinction between the *data* and *quæſita* of a problem.

60. Problems are divided into *determinate* and *indeterminate*, of which ſhe gives inſtances from Geometry. But in this Section ſhe treats only of ſuch as are determinate.

61. Here it is explained how equations are formed, from the dependance of quantities upon one another, whether they are known and given quantities, or unknown and required. The inſtances are taken from the properties of lines and figures.

62. How we are to argue from the given conditions of the queſtion till we come to an equation between the quantities given and required. This is explained geometrically, and by an abſtract arithmetical queſtion.

63. No more given quantities are to be aſſumed than are neceſſary, when they can be expreſſed by the known properties of the figure.

64. It will often happen, that the lines given in a figure are not ſufficient for forming the equations; then ſuch other lines muſt be drawn as may complete the figure, and bring us to a determination. A problem is propoſed to illuſtrate this; and the Propoſitions of *Euclid* are enumerated, which will be of uſe for ſuch purpoſes.

65. Here the Author propoſes and ſolves three or four geometrical problems, to ſhow the method of arguing from one condition to another, in order to obtain a final equation.

66. When the conditions of a problem involve the properties of angles, they muſt ſomehow be reduced to the properties of lines. This is exemplified in the problem of finding an equicrural triangle, in which either of the angles at the baſe is double to the angle at the vertex: which is reduced to the linear problem, of dividing a line in extreme and mean proportion.

67. Having thus ſhown how to find equations from the given circumſtances of a problem, ſhe proceeds to the reſolution of theſe equations, or to the finding the unknown quantity, by means of various reductions. For this end ſhe

ſhe gives us four axioms. By the firſt, ſhe ſhows the uſe of tranſpoſing quantities at pleaſure from one ſide of an equation to the other; which may always be done without deſtroying the equation, only by changing the ſigns of the terms ſo tranſpoſed.

68. By the ſecond axiom ſhe ſhows how we may take away any fractions that ariſe in an equation, and ſo reduce the whole to integral terms.

69. And how, by the ſame, any term may be freed from it's co-efficient.

70. By the third and fourth axiom ſhe ſhows how equations may be freed from ſurds and radicals; and of all theſe reductions gives us a variety of examples.

71. Equations prepared for ſolution, and diſtributed into their terms.

72. Equations further prepared, by which the unknown quantity will be found equal to a combination of known quantities, and a ſimple equation will be ſolved entirely.

73. If any power of the unknown quantity is found equal to known quantities, then the root may be extracted on both ſides.

74. If the equation is an affected quadratick, it may be ſolved by completing the ſquare on one ſide, and then extracting the ſquare-root on each ſide.

75. In quadratick equations the ambiguity of the ſigns will ſupply two values of the unknown quantity, which may therefore be both poſitive, both negative, or one poſitive and the other negative, or both imaginary, according to the values of the known quantities. What is analogous to this difference of ſigns in geometrical figures, is here ſhown, and all is illuſtrated by examples.

76. The Author ſhows us here the uſe of impoſſible or imaginary roots of equations. For they are a ſure indication, that the queſtion (as now propoſed) is impoſſible, either by chance or deſign. And the ſame thing is to be concluded, when the final equation brings us to any abſurdity or contradiction. This ſhe ſhows in ſeveral inſtances.

77. And ſometimes we may be brought to an identical equation; which only ſhows that the point required may be any where in the given line, as by the example.

78. Equations

78. Equations and problems are diſtinguiſhed into degrees, according to the dimenſions of the unknown quantity contained in them. Alſo, thoſe problems are called *Plane*, the reſolution of which requires only the ordinary Elements of Geometry. But if they require the deſcription of the Conic Sections, or other curves, they are *Solid Problems.*

79. Equations are not always of that degree which their higher powers ſeem to inſinuate, but may often be brought to a lower degree by an eaſy reduction: As by the examples.

80. Sometimes neceſſity, and ſometimes conveniency, will require, that more than one unknown quantity may be introduced in a problem; in which caſe (if the problem is determinate,) as many equations muſt be found as there are unknown quantities aſſumed. Then theſe are to be eliminated one by one, till we finally arrive at an equation, in which there is only one unknown quantity. The way of doing this ſhe ſhows by an example.

81. This method of elimination may be made uſe of, not only in ſimple equations, but alſo in affected quadraticks.

82. Higher equations may ſometimes be reduced, by eliminating their greateſt powers. And when thoſe powers have not the ſame index, they may be reduced to ſuch as have. Of both theſe reductions the Author produces ſeveral examples.

83. If there be ſeveral ſimple equations including as many unknown quantities, they may be ſeverally eliminated, and reduced to one equation including only one unknown quantity, though the calculation will often be tedious.

84. If there are not as many equations to be found as there are unknown quantities, the problem will become *indeterminate*, and will allow an infinite number of anſwers. Of this ſhe produces examples.

85. But if the conditions to be fulfilled, or the equations, are more than neceſſary, they may be inconſiſtent with each other, and ſo the problem will become impoſſible; or ſome of the conditions may coincide with others, and ſo be ſuperfluous.

86. Having

86. Having laid this foundation for calculating with arithmetical or algebräical quantities; ſhe now does the ſame for calculating with geometrical quantities, or with lines and figures. She begins with the operations of Multiplication and Diviſion, or, what is the ſame thing, with finding ſuch ſimple proportions, or conſtructing ſuch ſimple equations, as will give the values of the quantities required expreſſed by lines.

87. The operations of addition and ſubtraction of lines, when thus found, will be very eaſy and familiar.

88. Hence, by ſubſtitution, any given letter, or letters, may be introduced; or a plane may be transformed into another with a given ſide, or a ſolid into another with one or two given ſides, &c. by which the conſtruction of ſimple equations will be much facilitated.

89. This reduction is eaſily extended to fractions, the numerators or denominators of which are complicate terms.

90. But, without dividing a fraction into ſeveral fractions, the method of transformation may often be preferable, as is ſhown by a variety of examples.

91. Here it is ſhown how lines may be found, that ſhall expreſs the value of any quadratick radical, by only finding geometrically a mean proportional between two given quantities: excepting the caſe when that value is imaginary or impoſſible.

92. But, to reduce radical quantities to this rule, there will often be occaſion to have recourſe to the method of transformation, as appears by the examples.

93. Any quadratick radicals may be conſtructed by a right-angled triangle, either alone or combined with a circle, without transformation; though ſome transformation will often be found convenient. This illuſtrated by various examples.

94. The foregoing rules may eaſily be applied to the conſtruction of any affected quadratick equation; but they may all be conſtructed after a more general manner. For this purpoſe the Author aſſumes a general affected quadratick equation, which ſhe diſtinguiſhes into four, according to the variety of

of their ſigns. Theſe ſhe conſtructs, one after another, by right-angled triangles and a circle, and exhibits the roots, both affirmative and negative, by right lines.

95. The ſame equations may be otherwiſe, and more eaſily, conſtructed, when the laſt term is not a ſquare, but a rectangle.

96. Hitherto the learned Author has been laying down the principal rules of the Art of Computation, whether arithmetical, algebräical, or geometrical; ſhe now proceeds, as ſhe tells us, to ſhow their uſe in the ſolution of ſome particular Problems, to the number of 15, with which ſhe concludes this Section. The firſt is purely arithmetical, and to be found in moſt Books of Algebra.

97. The ſecond Problem is alſo very common, and is about the motion of two bodies with given velocities, in various circumſtances, general and particular.

98. The next is the famous Problem of King *Hiero's* crown, in which *Archimedes* diſcovered the quantity of baſer metal mixed with the gold, and which gave the occaſion to his celebrated ευρηκα.

99. The next Problem is concerning the relation of two weights to each other, and is purely arithmetical. And theſe Problems hitherto have produced only ſimple equations.

100. Then we have a Geometrical Problem, which amounts only to a ſimple equation, and is therefore eaſily reſolved and conſtructed.

101. The next Problem is geometrical, which ariſes to a ſimple quadratick equation, which is there conſtructed, or reſolved, geometrically.

102. Then a Geometrical Problem, teaching to inſcribe a cube in a given ſphere; which amounts only to a ſimple quadratick equation, and is there conſtructed, and the conſtruction proved by a ſynthetical demonſtration.

103. A Geometrical Problem, or rather Theorem, concerning a ſecant drawn through two concentrical circles, ſo that the parts intercepted by the circumferences ſhall be equal. This being the property of every ſuch ſecant, the

ſolution brings to an identical equation, which is a proper caution how to manage ſuch Problems, and what concluſions we are to derive from them.

104. Another Geometrical, or rather Algebräical, Problem.

105. A Geometrical Problem.

106. A Geometrical Problem, in which the magnitude of angles enters the calculation.

107. A Geometrical Problem, with a ſynthetical demonſtration.

108. The Author gives us here a very notable Geometrical Problem, which is, two contiguous arches of a circle being given, and alſo their tangents, to find the tangent of their ſum. And this ſhe extends very artfully to the ſolution of a much higher and more general Problem, which is, any number of arches and their tangents being given, to find the tangent of their ſum. By the way ſhe gives us a general Theorem, for finding all the poſſible combinations of any number of quantities given. She concludes with giving a general canon, or formula, for finding the tangent of any multiple or ſubmultiple arch; as alſo, ſhows the converſe of this Theorem.

109. Then we have a Geometrical Problem, which is, to find a triangle, the ſides of which and the perpendicular are in continued geometrical proportion. This amounts to a high equation, but is reduced to an affected quadratick: which is geometrically conſtructed.

110. The laſt Problem is that famous geometrical one, of triſecting a given angle. This ſhe divides into three caſes, according as the given angle is right, obtuſe, or acute. The firſt caſe ſhe ſolves by a ſimple quadratick equation, of which ſhe alſo gives us the conſtruction. The ſecond and third caſes ariſe to cubic equations, which ſhe reſerves till ſhe comes to treat of thoſe equations.

BOOK

BOOK I. SECT. III.

Of the Construction of Geometrical Places, and of Indeterminate Problems not exceeding the second Degree.

111. In this article the Author explains the nature of variable quantities; that there muſt always be two of them, at leaſt, in an indeterminate Problem, which are varied according to a conſtant law, which is expreſſed by a given equation.

112. A *Locus Geometricus* is a right line, or a curve, the *abſciſs* and *ordinate* (or the *co-ordinates*) of which are variable right lines, which in all caſes expreſs the variables of the equation. The abſciſs begins from ſome certain point taken at pleaſure in an indefinite right line, and the ordinate is placed at the end of the abſciſs, at a given angle. When a definite value is aſſigned to one of theſe lines, the curve, or locus, will give the definite and relative value of the other, agreeably to the equation: as by the inſtances may be ſeen.

113. Different equations will require different *loci*, and *vice verſâ*. And as the equations are of different degrees, ſo will the loci be alſo.

114. Of a ſimple equation the locus will always be a right line.

115. When any combination of the variables, in any one term, does not exceed the ſecond degree, the equation will always require a conic ſection for it's locus.

116. Theſe loci are here diſtributed into their ſeveral orders.

117. All equations of the firſt order, or which can belong to a right line, are here conſtructed.

118. In ſimple equations, ſometimes a determinate problem may be propoſed as an indeterminate, in which caſe one of the variables will vaniſh out of the equation, or not at all appear in it. Then the locus of the equation will be a

 right

right line, either perpendicular or parallel to the abſciſs. Of this the Author produces an inſtance or two, with their conſtruction.

119. The Author goes on to the circle, as the ſimpleſt curve, of which ſhe exhibits the firſt and ſimpleſt equations, whether we take the beginning of the abſciſs from the centre, or from the end of the diameter; and ſhows what the radius muſt be, in caſes not ſo ſimple: and tells us likewiſe when the circle will be only imaginary.

120. She proceeds then to the parabola, as the next ſimpleſt curve, of which ſhe exhibits the primary equations, whether the parameter be ſimple or complicate, whether the parabola be internal or external.

121. The next conic ſection is the hyperbola, or rather the two oppoſite hyperbolas, of which ſhe exhibits the ſimpleſt equations, when the ordinates are referred to the axis; whether the abſciſs commences from the centre, or from either of the vertices; or whether the equation is expreſſed by the axes, or by the parameter. She finds the equation when the hyperbola is equilateral; and reduces complicate parameters, or diameters, to ſimple ones.

122. She ſhows likewiſe what will be the ſimpleſt equation belonging to the hyperbola between it's aſymptotes.

123. The ſimpleſt equations are alſo derived for the ellipſis, whatever is the angle of ordination; and whether the abſciſs begins from the centre, or from either of the vertices; or whether the equation is expreſſed by the diameters, or the parameter. And what will be the equation, when the diameters and parameter are equal. In this laſt caſe, if the angle of ordination is a right angle, the ellipſis will degenerate into a circle. Complicate diameters and parameters are reduced to ſimple ones, as before in the hyperbola, from the equations of which thoſe of the ellipſis will differ only in their ſigns; ſo that they will eaſily paſs into each other.

124. When the ſimple equations to the diameters of the hyperbola, or ellipſis, are not given exactly in the terms of the diameters, but rather in diſguiſed terms; the Author ſhows how, by the Rule of Proportion, thoſe diameters may be found. Of which reduction ſhe gives Examples.

125. Or

125. Or when the ſame equations are expreſſed by parameters, though ſomething obſcurely; ſhe ſhows us how to find thoſe parameters, and gives Examples of it.

126. Having thus exhibited the ſimpleſt equations belonging to the Conic Sections, and ſhown how we may find the diameters or parameters when involved, by which theſe ſections may be deſcribed; the Author proceeds to conſtruct any complicate equations that may be given, belonging to theſe ſections or curves; in order to which, ſhe diſtributes all ſuch equations into three ſpecies or claſſes. The firſt are thoſe that contain the ſquare of one of the variables, and the rectangle of the other into a conſtant quantity. The ſecond ſpecies contains the rectangle of the two variables, with other ſimple terms. The third contains the rectangle and both the ſquares of the variables, with any other ſimple terms.

127. She then proceeds to conſtruct equations of the firſt ſpecies, however complicate they may be, and reduces them to a ſimple form, by one or two ſubſtitutions of new variables. And of this ſhe gives us two Examples. In the firſt, by one ſubſtitution, ſhe reduces the given equation to the ſimpleſt form belonging to the parabola, which ſhe then conſtructs. In the ſecond, ſhe reduces the given equation, by two ſubſtitutions, to the ſimpleſt form belonging to the hyperbola between the aſymptotes, which ſhe then conſtructs, and purſues it through all it's varieties. When the conſtant quantities are ſuch, as not to admit of theſe ſubſtitutions, ſhe changes them, by the tranſmutations ſhe had taught before, into ſuch as will be fit for thoſe ſubſtitutions.

128. Then ſhe reduces equations of the ſecond ſpecies to the firſt, by a method not unlike that of extracting the ſquare-root of an affected quadratick equation. By which means, and by a ſubſtitution, ſhe introduces a new variable. Of this ſhe gives an Example in an equation to the parabola, which ſhe reduces and conſtructs. Alſo, another to the hyperbola, reduced by two ſubſtitutions.

129. Then ſhe ſhows, by an example, how an equation of the third ſpecies may be reduced to the firſt, and ſo conſtructed.

130. Here

130. Here ſhe propoſes various complicate examples, of which ſome are to the parabola, ſome to the hyperbola, and ſome to the ellipſis, which require ſeveral ſubſtitutions and tranſformations; but are all reduced to ſimple equations, and conſtructed with great art and ingenuity.

131. All the variety of equations to the hyperbola between the aſymptotes, are reduced to four general equations, which are here conſtructed, by one, two, or more ſubſtitutions, or changing of the variables; and that according to all the variety of their ſigns. To illuſtrate theſe conſtructions, and to ſhow their application in particular caſes, ſhe propoſes and reſolves the ſeveral Problems following.

132. The equation of the firſt Problem is found to belong to the parabola, being the property of the focus of the parabola in reſpect of the directrix, which is therefore eaſily conſtructed by one ſubſtitution.

133. The equation of the next Problem is found to be a locus to the hyperbola between the aſymptotes, and is conſtructed by means of two eaſy ſubſtitutions.

134. This Problem is propoſed concerning the properties of two circles and their tangents, but the general ſolution and conſtruction of the equation require all the three conic ſections, according to the three caſes included in it. Theſe caſes are conſtructed ſeparately, by the help of ſeveral ſubſtitutions and tranſmutations.

135. A Problem to the three Conic Sections, according to it's three different caſes.

136. A general Problem ſolved by a canonical equation, and illuſtrated by three Examples of particular curves, of which the laſt ariſes to a cubical equation, and therefore goes beyond the Conic Sections.

137. A Problem concerning two equal interſecting circles, which ariſes to an equation to an ellipſis, which is here conſtructed by means of one ſubſtitution.

138. A Problem, or rather two Problems to the circle, with ſynthetical demonſtrations of the ſolution.

139. A

139. A Problem of a normal fliding between the fides of a right angle, and with one end defcribing a curve. This curve, by it's equation, is found to be an ellipfis, and is here conftructed.

140. The equation of this Problem is either to the parabola, the hyperbola, or the ellipfis, according to different circumftances, and is refolved by various fubftitutions, or changes of the indeterminate quantities, and is here conftructed.

141. The Method of Majority and Minority is here occafionally explained, which proceeds in the fame manner as the reduction of equations. For, by a feries of comparifons duly made, we may know which of two quantities is the greater or leffer.

142. A Problem producing an equation to the hyperbola between the afymptotes, which is very artfully refolved and conftructed, by three fubftitutions, or changes of the variable quantities.

143. Here the Author concludes her Problems, and recommends the proving the folution, after it is finifhed, by tracing back the feveral fubftitutions, and fo returning to the original equation. Of this fhe gives us two Examples in the foregoing Problems.

BOOK I. SECT. IV.

Of Solid Problems and their Equations.

144. THE Author having thus difpatched what are called Plane Problems, or fuch as require only equations of two dimenfions; fhe proceeds to thofe called Solid Problems, which require equations of more than two dimenfions, and therefore higher and more difficult conftructions. She begins by informing us what are the roots of fuch affected equations, or what are the values of the unknown and indeterminate quantities, which are to be extracted out of thefe equations. That they are fuch numbers or quantities, that, if they were to be fubftituted in the equation given, inftead of the root, they would reduce the whole to nothing; which would be a full proof, when the root, or roots, are extracted, that they are the true roots of the equation.

145. Or,

145. Or, in another acceptation, thofe fimple equations are often called the Roots of a compound equation, which, being multiplied into each other continually, will produce the equation given. Confequently that equation may be refolved into it's components by continual divifion. Hence every equation will have fo many roots as it has dimenfions. Of this fhe gives us inftances in equations of two, three, or four dimenfions, or of quadratick, cubick, and biquadratick equations, which are formed by the multiplication of fimple, but general equations, and which therefore will be the roots of the equations fo formed.

146. Hence, when any of the roots of a compound equation happen to be known, we have a method, by divifion, of depreffing that equation, and reducing it to a fimpler, which fhall include only the unknown roots.

147. From this way of raifing compound equations by multiplication, we may know the conftitution of every fingle term, when the whole equation is difpofed in a proper and regular order, and made equal to nothing. For the higheft term muft always be pofitive, and have no other co-efficient but unity, which can always be effected. The co-efficient of the fecond term will be the fum of all the roots, under their proper figns. The co-efficient of the third term will be the fum of the products of every pair of roots, &c. And the laft term will be the product of all the roots, affected by their proper figns.

148. It follows from hence, that, if the fecond term is wanting in any equation, then the fum of the pofitive roots will be equal to the fum of the negative; therefore, when that term is prefent and affirmative, the fum of the pofitive roots will be lefs than the fum of the negative; but the contrary, if that term be negative.

149. When any term is wanting in an equation, it's abfence is commonly indicated by putting an afterifm in it's place.

150. If no imaginary root appears in the equation, yet it may have them, two by two, always in pairs, and with contrary figns. If the degree of the equation is odd, it will have, at leaft, one real root; and if it's degree is even, it may have all it's roots imaginary. The like may be obferved of furd roots.

151. Many

151. Many ufeful indications, concerning the roots of an affected equation, may be had from the figns of the feveral terms.

152. A proof that, in cubick and biquadratick equations, if the fecond term is wanting, and the third term is pofitive, there will neceffarily be imaginary roots.

153. In any equation the affirmative roots may be made negative, and the negative affirmative, only by changing the figns of thofe terms which are in even places. Here the afterifm, or vacant place, muft always be reckoned for one. This proved by Examples.

154. The roots of affected equations may be increafed or diminifhed by any quantity at pleafure, by refolving the root into two parts, one unknown, and the other known; and that only by a fubftitution of equivalents. The new equation fo found will have the fame roots as the given equation, only they will be increafed or diminifhed by a known quantity. See the Author's Examples.

155. By a like fubftitution of equivalents, the roots of any equation, though unknown, may be multiplied or divided by a given quantity, and undergo many other changes at pleafure.

156. The reafon of thefe feveral proceffes is, that, as equals are always fubftituted for equals, fo the refults muft always come out equal.

157. The ufes of thefe fubftitutions are many. One of which is, that, though the roots of an equation are unknown, yet, by fuch a transformation, they may often become known.

158. Another ufe is, the freeing an equation from fractions or furds. Of this the Author produces feveral Examples.

159. Some neceffary conditions in the equation, in order to it's being freed from furds or radicals.

160. But the chief ufe of this tranfmutation of equations, is intirely to take away the fecond term from any equation by an eafy fubftitution: of which the Author gives feveral inftances.

161. Or the third term may be taken away, by ſolving a quadratick equation, the fourth by a cubic, &c.; as may appear from the Author's general proceſs.

162. In an equation wanting the ſecond term, the penultimate term may be taken away; but it will be by reſtoring the ſecond term.

163. Thus, in an equation wanting the third term, the ante-penultimate term may be taken away; and ſo on.

164. Or any equation, in which any term or terms are wanting, may be made complete by a new ſubſtitution.

165. If equations have diviſors of one, two, or more dimenſions, they are properly of that order, to which they may be reduced by diviſion.

166. Diviſion ought firſt to be tried by a diviſor of one dimenſion, then by thoſe of two, &c.

167. Equations of the third degree, if reducible, may be reduced by a linear or ſimple diviſor, which is to be found in the manner taught in the 56th Article before. If an equation of the fourth degree cannot be reduced by a diviſor of one dimenſion, to be found in the ſame manner, the reduction muſt be attempted by a diviſor of two dimenſions. To perform which, the Author throws out the ſecond term of the equation, as ſhown before, and then aſſumes two general equations of two dimenſions, and multiplies them together, and compares the terms of the produced equation with thoſe of the equation given. By this compariſon ſhe determines the co-efficients of the aſſumed equations, the laſt compariſon of which amounts to an equation, which in effect is no more than cubical. This cubic equation is reſolved by the Method of Diviſors, and it's roots, being ſubſtituted in the aſſumed equations, will make them become diviſors of the biquadratick equation propoſed. Of this method of ſolution ſhe gives us two Examples.

168. Here is the ſame proceſs as before, but after a more general manner, and applied to a particular biquadratick equation, which is reſolved by it.

169. Sometimes this method will ſucceed only by taking away the ſecond term of the equation, which will depreſs it to a quadratick.

170. The

170. The ſame method is purſued, but without taking away the ſecond term of the given biquadratick equation. Two general quadratick equations are aſſumed, and multiplied together, and the general co-efficients of the product are determined and eliminated, as far as may be, by a compariſon with thoſe of the given equation. The laſt co-efficient in theſe compariſons muſt be determined by the foregoing Method of Diviſors. But this way of reſolution ſeems to be too tentative to be of any general uſe. It is illuſtrated by three Examples.

171. The ſame method is carried on to equations of five dimenſions, in which the two aſſumed general equations are, one of two and another of three dimenſions. When, by compariſon, the general co-efficients are determined, they are ſubſtituted in the ſimpleſt of the aſſumed equations, which then becomes a diviſor of the given equation; as by two Examples.

172. The Author extends this method to equations of ſix dimenſions, which ſhe manages with great ſagacity and ſucceſs, though it muſt be owned to be very tedious, precarious, and tentative; but, however, is the beſt that can be had in theſe high equations. She aſſumes two general and ſubſidiary equations, one of two, and another of four dimenſions, which are multiplied together to produce a general formula for equations of ſix dimenſions, that may be reſolved into two ſuch equations. Then the general co-efficients are determined as before, and ſubſtituted in the ſimpleſt of the aſſumed equations, which will then become a diviſor of the given equation. Of this reduction ſhe gives us an Example.

But an equation of the ſixth degree may poſſibly be reſolved into two cubic equations, and not otherwiſe. She therefore aſſumes two general cubic equations, and multiplies them together, to conſtitute a general formula for theſe equations. Then, a particular equation of ſix dimenſions being given, the general co-efficients are determined by compariſon, as far as that can be done, and their values are finally ſubſtituted in one of the aſſumed equations, in order to form a diviſor to the given equation.

173. The Author aſſures us, that the ſame method might be applied to the ſolution of higher equations, if it was not for the exceſſive tediousneſs of the operations. It may very well be ſuppoſed, that the calculation will become

very laborious in thofe equations, by what we fee in thefe of a lower order. And as the method is but tentative at beft, it can hardly deferve to be profecuted any further; efpecially as we have an *exegefis numerofa* to recur to in thefe cafes, which, though only an approximation to the root, yet will anfwer all real occafions that can offer. The Author now proceeds to propofe and refolve fome particular Problems, in order to fhow the ufe and application of what is now delivered.

174. The firft Problem is purely arithmetical, and is elegant enough: *To find four numbers which exceed one another by unity, and their product is* 100. The equation of this Problem amounts to a biquadratick equation with all it's terms; but, by throwing out the fecond term, it is reduced to a quadratick with four roots. Thefe are irrational, of which two only are real, one pofitive, the other negative, either of which will folve the Problem. The firft and leaft of the four numbers required, when reduced to a decimal, will be the negative number.

175. The next is a Geometrical Problem, relating to a right-angled triangle. It's equation is a biquadratick with all it's terms, but when the fecond term is taken away, it degenerates into a quadratick with a plane root, but irrational.

176. A Geometrical Problem producing a biquadratick equation, the four roots of which are irrational, and may be all real, and are exhibited by the figure.

177. An equation may often appear of a higher order than the Problem really requires, if a prudent choice is made of the unknown quantity, by which the Problem is determined. This is illuftrated in feveral appofite Examples.

178. Another artifice that often prevents Problems from rifing to too high equations, is finding two values of the fame unknown quantity, and making them equal. An inftance of this is feen in the next Problem.

179. This Problem is, *in a given circle to infcribe a regular heptagon.* The Author gives feveral folutions of this Problem, which amount to high equations; but, by being compared with each other, are reduced lower. At laft fhe brings it to a cubic equation with a plane root. This is performed by finding two different expreffions for the fame quantity, and comparing them together.

180. When

180. When cubic (or higher) equations cannot be thus reduced, their roots may be found analytically, but involved in furds, by what are called *Cardan's* Rules. But the geometrical method will be more univerfal, by conftructing them, and finding their roots by the interfection of curve-lines.

181. She begins with the analytical folution, or with finding *Cardan's* Rules: All cubic equations, that want the fecond term, are reprefented by four general formulæ, differing only in the feveral changes of the figns. To refolve the firft general formula, fhe divides the unknown root into two parts, which, after fubftitution, gives room for fplitting the equation into two, fuch as may eafily be refolved feparately. This finds commodious values for the two affumed parts of the root, and brings us two cubical radicals for the value of the root. See the Philofophical Tranfactions, Number 309.

182. The folution of the fecond formula does not differ from the firft, but only in the figns.

183. The fame may be faid of the third.

184. And likewife of the fourth.

185. All the four formulæ are folved fomething differently, in which the two parts of the root have only one cubic radical; but which coincide with the foregoing folution, and are eafily reduced to it.

186. The limits of thefe roots are here affigned, and it is fhown when they will be real, and when two of them will be only imaginary.

187. When one root is found of a cubic equation, the other two may be found without divifion. For, as unity itfelf has three cubic roots, fo any other quantity has the fame. Therefore, multiplying the root found by the three roots of unity fucceffively, we fhall have the three roots of the given equation. This is proved here fynthetically, by returning to the original equation. See Phil. Tranf. No. 309.

188. This method of folution is illuftrated, by applying it to a given cubic equation, of which the three roots are thence found.

189. Or, without recurring to the general ſolution, any particular cubic equation may be ſolved, by purſuing the method of that ſolution. Of theſe here are given ſeveral Examples.

190. The Author proceeds to the ſolution of biquadratick equations, of which ſhe takes a general formula, with the ſecond term abſent. Then aſſumes two general quadratick formulæ, which, multiplied together, produce a general biquadratick equation; and, by compariſon with the firſt general equation, ſhe determines the aſſumed co-efficients. This will bring her to a transformed cubic equation, in the manner taught in Article 167 aforegoing. And thus ſhe proceeds to determine the four roots of the aſſumed biquadratick equation. See Phil. Tranſ.

This ſolution ſhe applies to an Example.

191. From the algebräical reſolution of theſe equations, ſhe proceeds to the more general (as ſhe calls it), or to the geometrical ſolution, which is, by conſtructing the ſeveral *loci geometrici*, or curve-lines, adapted to every equation conſiſting of two indeterminates. Every determinate equation may be reſolved into two indeterminate equations, by introducing a quantity into it at pleaſure. Theſe two equations muſt conſiſt of the ſame two variable quantities, and the ſame conſtant quantities, and may be conſtructed by two curves. If thoſe two curves are combined in ſuch manner, as that they ſhall have a common abſciſs, they will alſo have ſome common ordinates at their common points, that is, their points of interſection. Theſe common ordinates will be the roots of the determinate equation, if the quantity repreſenting thoſe roots is made one of the variable quantities. To exemplify this, ſhe aſſumes a determinate biquadratick equation, and alſo an equation to the parabola. This ſhe introduces, by ſubſtitution, into the given biquadratick, which will then be an indeterminate equation to the hyperbola. She then conſtructs theſe two curves upon a common axis, and draws four ordinates from the four points of interſection of the curves, which will be the roots required.

192. From this conſtruction theſe notable circumſtances will evidently follow; that the poſitive and negative roots will be on different ſides of the common abſciſs; that, when two ordinates become equal, or when the two curves do not

cut

cut but touch each other, two roots of the equation will be equal; or, when the two curves cut each other in the vertex, one of the roots will be equal to nothing; and where the curves neither touch nor cut, the roots will be impoſſible.

193. It is here ſhown, that, as there may be great variety in reducing a determinate equation given, to two indeterminate equations, in order to be conſtructed; ſo ſuch a choice is to be made of the two *loci*, that the conſtruction may be as ſimple as poſſible. According as the equation is in degree, ſo each *locus* ſhould be taken, as together to make up nearly the dimenſions of the given equation.

194. Here it is ſhown, by an Example, how the ſeveral *loci* to the Conic Sections are to be diſtinguiſhed from one another.

195. Other cautions to be obſerved, in adapting the *loci* to their equations.

196. Here follow ſome Examples, to illuſtrate the foregoing doctrine. The firſt is, a determinate cubic equation wanting the ſecond term, which is reduced to a biquadratick, by multiplying the whole by the root, and a ſimple equation to the parabola is aſſumed. This is introduced into the given equation by ſubſtitution, by which it becomes an indeterminate equation to the circle. Then theſe two *loci* are combined, or conſtructed to a common abſciſs, and from their interſection a common ordinate is drawn, which will therefore repreſent the root of the given equation. Their other interſection is at the vertex, and therefore it's root will be nothing, which was introduced into the equation. The truth of this conſtruction is confirmed by a demonſtration.

197. The ſame equation is again conſtructed by combining two parabolas, and the conſtruction demonſtrated.

198. Or, to conſtruct the ſame equation, the equilateral hyperbola might be introduced, only by ſubtracting one of the equations to the parabola from the other.

199. Or, laſtly, by a ſmall alteration, one of the *loci* might have been to the circle, the other to the hyperbola.

200. But,

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200. But, without increaſing the dimenſions of the cubic equation, it may be conſtructed by an hyperbola between the aſymptotes, combined with a parabola; as is here performed, and the conſtruction demonſtrated. And ſo may all other equations be conſtructed, that do not exceed the third degree.

In her next Example ſhe takes a determinate equation of the fourth degree, which ſhe changes into an indeterminate, by the ſubſtitution of an equation to the parabola. Into this ſhe introduces an equation to the circle, and then conſtructs it by means of theſe two *loci:* which conſtruction ſhe then demonſtrates.

For another Example ſhe takes a determinate cubic equation, into which ſhe introduces a known root by multiplication, which raiſes it to a biquadratick. Then taking an equation to the parabola, by the ſubſtitution of this after various manners, ſhe produces ſeveral indeterminate equations; the laſt of which, being to the circle, ſhe makes choice of for conſtructing the biquadratick equation. One of it's roots is the known root that was introduced, two are imaginary, and the fourth is a real but negative root. Then ſhe demonſtrates the conſtruction.

Another Example is, an equation of ſix dimenſions, but, being diviſible by a diviſor of two dimenſions, it is reduced to a biquadratick equation. By various ſubſtitutions of an equation to the parabola, various *loci* are formed, of which ſhe conſtructs one, which is to the equilateral hyperbola. But theſe two *loci,* being combined as their equations require, will no where interſect each other, or will have no common ordinates. Which proves, that all the roots of the given equation are imaginary and impoſſible.

201. In this Example a biquadratick, or cubic, equation is propoſed, to be conſtructed by two conical *loci,* not to be found (as before) from the given lines of the equation, but ſuch as are already known and deſcribed, or otherwiſe by ſuch as ſhall be like to theſe. This is performed by deriving the two *loci* in general (as before), and then introducing new quantities, which are to be determined from the known lines of the given *loci,* according to their various circumſtances. This equation, therefore, is conſtructed by means of a given parabola, combined with a given hyperbola.

It

If it ſhould be required to conſtruct a biquadratick equation with a given parabola, and with an ellipſis that is of the ſame ſpecies with an ellipſis given; here is an inſtance of it, by means of introducing new quantities into the equation; which are afterwards to be determined as occaſion ſhall require. And the truth of the conſtruction is demonſtrated at length.

202. The Author here, by way of anticipation, gives us ſome conſtructions of equations that exceed the fourth degree, though ſhe reſerves the fuller treating of ſuch conſtructions to her next Section. She aſſumes a determinate equation of the fifth degree, and likewiſe an indeterminate equation to the parabola, and, by ſubſtitution, forms an equation, or locus, to a line of the third degree, which, combined with the parabola, will conſtruct the given equation. Or, ſhe ſhows how it may be done with the ſame locus combined with an hyperbola. Or, with an hyperbola, and the firſt cubic parabola. Likewiſe, ſhe conſtructs an equation of the ſixth degree, by a parabola combined with a line, or locus, of the third degree: of which equation ſhe finds two real roots, one affirmative and the other negative, and the other four are imaginary.

203. Then ſhe tells in what order the *loci* muſt riſe, by which we would conſtruct higher equations; and conſtructs (for example) an equation of eight dimenſions by means of a parabola, combined with another locus of four dimenſions.

204. She then obſerves, how equations of the ninth degree (and therefore thoſe of the eighth degree, reduced to the ninth by multiplying by the unknown root,) may be conſtructed by combining two *loci* of the third degree: which rule ſhe makes general.

205. The moſt natural way of conſtructing an equation of any degree, is by a right line for one of the *loci*, and a curve of the ſame degree for the other. As an example of this method, the Author aſſumes a definite equation of the fifth degree, makes one of the diviſors of the laſt term to become indefinite, that is, aſſumes a locus to a right line, and, ſubſtituting it in the given equation, makes it become an indefinite equation of the ſame degree as the equation given. This being conſtructed, and the right line drawn as it ought to be by

the nature of the equation, the common ordinates will determine ſo many abſciſſes, which will repreſent the roots of the given equation. Thoſe roots will be impoſſible, where the right line does not meet the curve.

206. She tells us this method may be of uſe in verifying other conſtructions; then proceeds to particular Problems, with their conſtructions.

207. The firſt is a Geometrical, or rather Analytical Problem; *between two given quantities, to find as many mean proportionals as we pleaſe.* This is applied to finding two mean proportionals, and ariſes to a ſimple cubic equation, which ſhe raiſes to an affected biquadratick, by multiplying it by the unknown root. Then aſſumes a locus to the parabola, and, by ſubſtituting it various ways in the given equation, ſhe forms ſeveral other *loci*, one to a parabola, one to an hyperbola, and one to a circle. This laſt ſhe combines with the aſſumed locus to the parabola, and conſtructs the equation given; finding one real affirmative root, and the root that was introduced which is equal to nothing, and the other two roots will be imaginary.

208. Or, without introducing a new root equal to nothing, ſhe conſtructs it by a parabola, and an hyperbola between the aſymptotes.

209. To find three mean proportionals is a plane Problem.

210. To find four mean proportionals amounts to a ſimple equation of the fifth degree, which ſhe conſtructs by means of a parabola combined with an hyperboloid of the third degree.

211. Or, by the common hyperbola between it's aſymptotes, and the ſecond cubical parabola.

212. To find five mean proportionals amounts only to a cubical equation. Then ſhe obſerves, by what *loci* fix, ſeven, or any other number of mean proportionals may be found.

213. The next is a Geometrical Problem, *of three contiguous chords being given, terminating at the diameter of a circle, to find that diameter*; which Problem has two caſes. For the middle chord may cut the diameter, either within the circle or (produced) without. The equation that ariſes for the ſolution of this

Problem is cubical, which ſhe multiplies by the root to make it a biquadratick. Then, aſſuming a locus to the parabola, by ſubſtitution ſhe finds another locus, which is to the circle; by the combination of which two *loci* ſhe finds the three roots, and then determines which of them will ſolve the preſent Problem. After which ſhe proceeds to the other caſe, which, with little variation, requires the ſame conſtruction.

214. A Geometrical Problem, by which the Problem of § 176 is made more general, the equation aſcending to the fourth degree. It is conſtructed by a parabola combined with an hyperbola.

215. This Problem is, *to triſect a given angle*, (ſee § 110.) and amounts to a cubic equation, which is conſtructed by two *loci*, the parabola, and the hyperbola between it's aſymptotes. The conſtruction is demonſtrated, and extended to all the caſes.

216. A further explanation of the triſection of an angle, ſhowing how the three roots of the equation ſerve for all the three ſeveral caſes, which are implied in the triſection of any angle.

217. The ſame otherwiſe conſtructed, by combining two other *loci*, one to the parabola, and the other to the circle.

218. This Problem of dividing a given arch into any given number of parts, is here extended to five equal parts, and ariſes to an equation of the fifth degree. It is conſtructed by aſſuming a locus to the parabola, and thence forming an indeterminate equation of the third degree, which is conſtructed by a curve proper to it. Theſe two, being combined, give all the five roots of the equation.

219. And this may be extended to the dividing any angle into any greater odd number of equal parts.

BOOK I. SECT. V.

The Construction of Loci exceeding the second Degree.

220. HAVING discoursed at large of the use of the Conic Sections, as geometrical *loci* for the construction of equations; the Author proceeds now to higher curves, and their description, as the proper *loci* for constructing equations of more dimensions. These curves, she says, may be described in two different manners; one is, by finding as many points as we please in each curve, and tracing regular curves through them. The other is, by taking a curve already described of a lower order, and finding by that the points of the other curve, or locus.

221. In order to describe a curve by an infinite number of points, from it's equation we must derive the value of one of it's unknown quantities, and suppose it the ordinate of a curve. Then we must assume a succession of values of the other unknown quantity, or the abscis, and then the corresponding ordinate will become known, and so give us a succession of points in the curve, through which we may trace a regular curve, which will be one locus. Of this she proposes an Example in an equation of three dimensions.

222. This ordinate may be drawn at any constant angle to it's respective abscis.

223. As an example of this description of a curve by points, the Author assumes the equation to an equilateral hyperbola; and, interpreting the abscis by small numbers continually, she finds the corresponding ordinates, which give so many points in the curve.

224. And the same thing will obtain if the abscis is interpreted by negative numbers, beginning from the centre of the hyperbola; so that the same hyperbola will arise, but only in an inverted position.

225. And when the ordinate is made nothing, the value of the abscis will show when the curve cuts the axis.

226. Also,

226. Alſo, intermediate points may be found, by intermediate values of the abſciſs and ordinate.

227. A Rule to find whether a curve has aſymptotes or no, and where they are if it has any.

228. But this Rule holds only when the aſymptotes are parallel to the co-ordinates; for the hyperbola has it's aſymptotes, which may be found from another equation to the ſame curve, and by the ſame rule.

229. The affair, of finding the aſymptotes of curves, properly belongs to the Method of Infiniteſimals, to which therefore it is referred.

230. Other circumſtances of the propoſed curve are here inquired into, as, whether it is convex or concave towards it's axis. This is eaſily determined by the Rule of Proportion. For, if a triangle is inſcribed in the curve, and an ordinate is drawn which is in common both to the curve and the triangle; if the ordinate to the triangle is leſs than that to the curve, the curve will be concave to it's axis; otherwiſe not.

231. But this Rule will not always obtain in all curves; for, in ſome, particular methods muſt be uſed, as will be ſeen hereafter. The Author proceeds to give another Example of deſcribing curves by points, which is the firſt cubical parabola. Of this ſhe determines a ſufficient number of points, to ſhow it's progreſs, that it cuts the axis only in one point, that it goes on *ad infinitum*, that it has no aſymptotes, that it is concave towards it's axis, and that it has a negative branch like the poſitive, but contrarily poſited.

The next Example is of the firſt cubical hyperboloid, the form of which ſhe determines by finding it's points; as alſo it's aſymptotes, and other circumſtances.

She then gives an Example of a curve of the fourth degree, the form of which ſhe determines by finding the ſeveral points.

232. She further proſecutes the ſame equation through all it's varieties, of poſitive, negative, and imaginary roots; ſhowing the different circumſtances of the curve, and of it's ſeveral branches, which reſult from thoſe roots.

9 Another

Another Example of an equation of three dimenſions, from the roots of which, and finding the moſt material points, the form and other circumſtances of the curve belonging to it are determined: as it's aſymptote, it's conjugate oval, &c.

Another Example of a curve of three dimenſions, in which the principal points are determined by the ſeveral roots of the equation.

233. The ſame equation and the ſame curve is further proſecuted, and other of it's properties diſcovered: as it's two parts extending to infinity, their common aſymptote, the convexity towards it's axis, &c.

234. The ſame method, of deſcribing the curve by points, may be extended to equations in which the indeterminates are involved together, and not eaſily ſeparable. The points required may ſtill be found, though the trouble will be increaſed.

235. The Author makes an apology, for ſeeming to depart from the method ſhe had preſcribed to herſelf, in treating of theſe high equations and their curves; and then illuſtrates what ſhe has delivered, by propoſing and ſolving ſeveral Problems.

236. The conſtruction of the firſt Problem produces a well-known curve called the *Ciſſoid* of *Diocles*, and ariſes to an equation of the third degree. This locus the Author deſcribes, by finding ſeveral of the principal points, and determines it's aſymptote.

237. In this Problem the Author finds another curve by it's points, the equation of which ariſes to four dimenſions.

238. A Problem in which the Author conſtructs a curve, which ſhe calls the *Witch*. It's equation ariſes to three dimenſions, and ſhe determines it's aſymptote and other circumſtances.

239. The curve of the next Problem will be the *Conchoid* of *Nicomedes*, the equation of which ariſes to the fourth degree. This ſhe conſtructs by finding it's principal points, it's two diſtinct parts ſeparated by a common aſymptote, it's concavity and convexity, and that it has points of contrary flexure

flexure and regreſſion. This is in the firſt caſe; for ſhe diſtinguiſhes the Problem into three caſes, which ſhe purſues ſeparately.

240. As the firſt caſe depended upon the equality of two certain lines, ſo this requires that one of them ſhall be bigger than the other, and ſo will produce a different figure with ſomething different properties. The point of regreſſion in the former caſe now becomes a node, where the curve croſſes itſelf, and forms a foliate. The aſymptote remains as before, and the curve will have a like concavity and convexity towards it.

241. The other caſe is, when that line, which before was the greateſt of the two, is now the leaſt. This produces a great alteration in the curve of the former caſe; for now the foliate entirely vaniſhes, and makes the curve have a continued curvature at it's vertex, not much unlike that part on the other ſide of the aſymptote.

242. The Author propoſes a way here, of improving this method of deſcribing curves by points; which is by geometrical conſtruction. In this her firſt Example of it, ſhe reſumes the *Ciſſoid* of *Diocles* and it's equation, § 236, which ſhe conſtructs an eaſier way by geometrical effection.

In her ſecond Example ſhe reſumes the curve of § 237, which ſhe conſtructs after a like manner.

Then ſhe does the ſame by the curve called the *Witch*, § 238.

And by the *Conchoid* of *Nicomedes*, of § 239, which ſhe conſtructs geometrically in all it's varieties.

243. The foregoing conſtructions are eaſily performed by the aſſiſtance of a circle; others may be made by the help of other ſimple curves. As, here an equation of four dimenſions is conſtructed by means of a parabola; but that parabola muſt be varied for every new ordinate. However, every new parabola gives four points in the curve.

244. Parabolas are here enumerated, and diſtributed into orders, according to their dimenſions. There is only one of the firſt order, which is the *Apollonian*, or common parabola: two cubic parabolas, or of the ſecond order; three of the third order, or of four dimenſions; &c.

245. In

245. In thefe feveral orders of parabolas, thofe are called firft parabolas in whofe equation the abfcifs afcends no higher than to the root, or firft power. She begins with the conftruction of the firft cubic parabola, the equation of which fhe changes (by fubftitution) into that of the common parabola, which fhe conftructs; then, by means of this fhe eafily finds the points of the other parabola: and that both for the pofitive and negative branch.

246. The Author proceeds to conftruct the firft parabola of the fourth degree, by changing it's equation of four dimenfions (by fubftitution) into the equation of the firft cubical parabola, which has been conftructed. Then, by the help of fimilar triangles, for every ordinate of the affumed parabola fhe determines a point of the curve required, in each branch affirmative and negative.

247. By the fame method, from the firft parabola of the fourth degree the Author conftructs the firft parabola of the fifth degree, as to both it's branches affirmative and negative.

248. She then fhows, in general, that we may always conftruct a firft parabola of any degree, by means of a triangle, and of the firft parabola of the next lower degree.

249. The Author then proceeds to conftruct other parabolas befides the firft, and that of any degree, by means of the firft, which fhe fuppofes already defcribed. As, here fhe defcribes the fecond cubic parabola, by finding it's ordinate from that of the firft, being reduced to a common abfcifs. And, in like manner, fhe conftructs the third parabola of the fourth degree, by reducing the value of one ordinate to that of another.

250. She adds here a ufeful Remark concerning any of thefe parabolas, or paraboloids; which is, that the fecond parabola of the fourth degree is no other than the common parabola, only redoubled on the negative fide: and fo in all other, in which the index of the power of the ordinate is double to that of the abfcifs, and both even numbers. But if the index of the power of the abfcifs is an odd number, the curve will be no other than the common parabola, without fuch reduplication. And this holds good of all hyperbolas as well as parabolas.

251. She

251. She goes on to the conftruction of hyperbolas (or hyperboloids) of any degree. There are only two of the third degree; the firft has it's ordinates reciprocally proportional to the fquares of the abfciffes, in the fecond the fquare of the ordinate is reciprocally as the abfcifs. The firft of thefe fhe conftructs by the help of a common parabola and hyperbola, by means of which fhe finds it's points. The other will be the fame curve in effect, and may be conftructed the fame way, only by changing the co-ordinates into each other.

252. The Author proceeds to conftruct hyperboloids of the fourth degree, or fuch wherein the ordinate is reciprocally as the cube of the abfcifs; or the fquare of the ordinate is reciprocally as the fquare of the abfcifs; or the cube of the ordinate is reciprocally as the abfcifs. The firft fhe conftructs by the help of the common hyperbola and the firft cubical parabola; the fecond is no other than the common hyperbola itfelf; and the third is the fame as the firft, if the co-ordinates change places.

253. She goes on to conftruct hyperboloids of the fifth degree; and, firft, that in which the ordinate is reciprocally proportional to the fourth power of the abfcifs. She finds the points of this, by firft conftructing a common hyperbola, and then, in proper circumftances, a firft paraboloid of the fourth degree. She alfo conftructs another hyperboloid of the fifth degree, in which the fquare of the ordinate is reciprocally as the cube of the abfcifs, by affuming two other curves of an inferior degree. In all thefe conftructions fhe determines the afymptotes of the curves, and their other affections. And the fame method might be purfued in *loci* of higher degrees.

254. She obferves that all firft parabolas, defcribed about the fame axis, will cut one another in the fame point. This point will be diftinct from their common vertex; and, befides, they muft all have the fame parameter.

255. Likewife, that thefe firft parabolas, in tending to this common interfection, the higher their dimenfions are, the nearer they approach to the tangent; and, after they are paft it, the nearer they approach to the axis. And the firft hyperboloids have alfo a like property.

256. Having conftructed thefe paraboloids and hyperboloids, or curves of two terms only; the Author proceeds to fuch as have feveral terms, which fhe

diſtinguiſhes into three caſes. The firſt caſe is of thoſe curves, or their equations, in which the ordinate is but of one dimenſion only, and is found only in one term. In the ſecond, the ordinate ariſes to any power, but is found in one term only.

In the third, the ordinate is found in more terms than one, and of any number of dimenſions.

[257. She gives here an Example of the firſt caſe. The equation of the curve to be conſtructed is of the fourth degree, and has three terms. By a convenient ſubſtitution this equation is reſolved into two others, one of which contains only conſtant quantities, and the other belongs to a firſt parabola of the fourth degree, which is here conſtructed, and the co-ordinates of the other curve are eaſily derived from it; which curve, it is obſerved, will be a portion of a parabola of the ſame degree.

258. Another Example of the ſame caſe, in which the equation of the curve to be conſtructed has three terms and four dimenſions. Here again, firſt, the equation is reſolved into two others by a ſubſtitution, and then the curve is conſtructed by means of two firſt parabolas, one of three, and the other of four dimenſions.

259. A third Example of the ſame caſe. The equation of the curve is of four dimenſions, and has four terms. This likewiſe is reſolved into two other equations by a ſubſtitution, of which one is ſimilar to that which was conſtructed in the preceeding example, and the other is to the *Apollonian* parabola; and by means of theſe two curves the required one is eaſily conſtructed.

Here the Author remarks that, if an equation ſhould more abound in terms, the ſame artifice might be uſed; and that, although the conſtruction in this caſe might become more compounded and perplexed, yet the ſame method would obtain.

She obſerves alſo that the equation in this example might have been reſolved, by another ſubſtitution, into three equations belonging to as many parabolas of different orders; and then, by means of theſe auxiliary curves, the principal curve might have been conſtructed.

260. It

260. It is here ſhown, that the co-ordinates of theſe curves may make any angle.

261. The Author gives here an Example of the ſecond caſe, in the conſtruction of a general equation of many terms, which, by a convenient ſubſtitution, ſhe reduces to caſe the firſt. See the Example.

262. An Example of conſtructing a curve of the third caſe. The equation here propoſed is general, and is reſolved, by a proper ſubſtitution, into others which belong to the firſt caſe; ſo that, by the conſtruction of theſe curves, the co-ordinates of the propoſed curve are obtained.

263. Hitherto the Author has conſidered only thoſe equations which have their indeterminate quantities ſeparate; ſhe here obſerves that, when the indeterminates are involved with each other, the foregoing rules cannot take place, but that a ſeparation of the variable quantities muſt be made, either by common diviſion, or by the extraction of roots, or by a congruous ſubſtitution, or by other expedients. She then gives two examples of the ſeparation of the indeterminate quantities: in the firſt, it is performed by common diviſion; in the ſecond, by a convenient ſubſtitution.

264. Having ſhown how to prepare equations of that kind for conſtruction, ſhe proceeds to the actual conſtruction of them, taking here the firſt equation in the preceding article, and conſtructing it by means of equations which come under caſe the firſt of article 256.

265. The conſtruction of the other equation in § 263: which, it is ſhown, may be performed by caſe the third of § 256.

266. A remark, That a convenient ſubſtitution may be of uſe even in thoſe caſes in which the indeterminate quantities are already ſeparated; and may ſuggeſt a conſtruction which is more eaſy and elegant.

267. An inſtance of the truth of the foregoing remark appears here, where the conſtruction of a curve, the equation of which has four dimenſions, is

facilitated by a ſubſtitution, although the variable quantities in that equation were ſeparate. With this the Author ends her examples of the conſtruction of curves.

BOOK I. SECT. VI.

Of the Method De Maximis et Minimis, *of the Tangents of Curves, of Contrary Flexure and Regreſſion; making uſe only of the* Common Algebra.

268. THE Author here obſerves that, although the Calculus of Infiniteſimals* be the ſimpleſt and ſhorteſt method, and the moſt univerſal, for managing ſuch ſpeculations, yet the ſolution of ſuch queſtions may be performed by common Algebra, in curves that are expreſſed by finite algebräical equations.

She begins with the *Maxima* and *Minima*; that is, to find in geometrical curves the greateſt or the leaſt ordinates; and ſhows that, in either caſe, two ordinates coincide, and conſequently two abſciſſes become equal; and thence two roots of the equation belonging to the curve, taken either in terms of the letter which expreſſes the abſciſs, or of the letter which expreſſes the ordinate, become equal to each other.

Her firſt Example is, To find the greateſt or leaſt ordinate when the curve is an Ellipſis; which ſhe does by forming a quadratick equation which has equal roots, and comparing it, term by term, with the equation of the curve. She then ſhows how to perform the ſame thing when the equation of the curve is of the third, fourth or higher degree; which is, by forming an equation of the ſame degree, that has two equal roots, and comparing it, term by term, with the equation of the curve. See the Examples.

269. A ſhorter and eaſier way of doing the ſame thing; which is, by multiplying the terms of the given equation by the terms of an arithmetical

* Rather *Fluxions*. EDITOR.

progreſſion.

progreſſion. For, if an equation has two equal roots, (which is the caſe of a *maximum* or *minimum*,) one of theſe roots will, of neceſſity, be included in the product of that equation multiplied by the arithmetical progreſſion. This is demonſtrated; and the two preceding examples are reſumed, and the ſame reſults obtained, although different progreſſions are uſed.

270. The Author proceeds to find tangents and perpendiculars to curves by a like method; previouſly ſhowing that the queſtion is reduced to this: To find a circle that ſhall touch the curve in any given point. This alſo is performed by means of equations that have two equal roots: which ſhe explains, and illuſtrates by an example of drawing a tangent to the *Apollonian* parabola. The equation which thus ariſes is ſolved, firſt, by comparing it with another quadratick having two equal roots; ſecondly, by multiplying the terms of it by the terms of the arithmetic progreſſion 3, 2, 1; and, laſtly, by multiplying the terms by the progreſſion 2, 1, 0.

271. Another Example of drawing a tangent to a curve of which the equation is cubical, worked both by comparing it with an equation of the ſame degree which has two equal roots, and by multiplying the terms of it by the arithmetical progreſſion 3, 2, 1, 0.

272. It is obſerved, that, in general, the moſt convenient progreſſion will be that which forms the exponents of the letter according to which the equation is ordered.

273. The Problem of drawing tangents is ſolved in a way ſomewhat different, but more ſimple; and the formulæ here derived are of uſe alſo in finding points of contrary flexure and regreſſion.

274. Points of Contrary Flexure and Regreſſion are here defined; and it is ſhown that, as the nature of *maxima* and *minima*, and of tangents, requires equations that have two equal roots, ſo in contrary flexures and regreſſions three equal roots are required. An example of finding the point of contrary flexure is given, by way of illuſtration.

275. The Author obſerves that the operation is the ſame for finding the points of regreſſion in curves, as for finding points of contrary flexure; ſo

that, to diftinguifh them, there is no other way, but to find, by means of a conftruction, the figure and proceeding of the curve.

She fays that the fame ambiguity arifes in queftions *de maximis et minimis*, which can only be removed by acquiring fome knowledge of the difpofition of the curve. She then obferves that, by the fame condition of three equal roots, we may find the *radii* of curvature; but, intending to treat of thefe things in the fecond Volume, fhe here puts an end to the firft.]

N. B. It being my intention to deliver what I have to offer on the fecond Volume in Notes, as is mentioned in my Advertifement prefixed to this Work, the reader will fee the propriety of my continuing the *Plan of the Lady's Syftem of Analyticks* no further. J. H.

ANALYTICAL INSTITUTIONS.

BOOK I.

ANALYTICAL INSTITUTIONS.

BOOK I.

THE ANALYSIS OF FINITE QUANTITIES.

THE *Analyſis* of Finite Quantities, which is commonly called the *Algebra* of *Carteſius*, is a method of ſolving Problems by the uſe and management of finite quantities: that is, from certain quantities and conditions, which are given and known, we may come to the knowledge of others which are unknown and required; and that by means of certain operations and methods, which I propoſe to explain by degrees in the following Sections. Introduction. What is *Algebra*, or *Analyticks*.

SECT. I.

Of the Firſt Notions and Operations of the Analyſis of Finite Quantities.

1. THE primary operations of this Algebra, or Analyticks, are the ſame as thoſe of common Arithmetick; which are, Addition, Subtraction, Multiplication, Diviſion, and Extraction of Roots. But with this difference, that whereas in Arithmetick thoſe operations are performed with numbers, in Algebra they are performed (or perhaps only inſinuated) with ſpecies, or the letters of the alphabet; by which quantities are denominated and calculated in the abſtract, of whatever kind they may be, whether Geometrical or Phyſical, as Lines, Surfaces, Solids, Forces, Reſiſtances, Velocities, &c. And therefore this kind The operations of Algebra, what.

 of

of Arithmetick is often called *The Algorithm of Quantities*, or *Specious Arithmetick*. And indeed this is of a much more excellent and general nature than that can be, though all it's operations are the ſame; as well becauſe theſe quantities are not confounded one among another in their operations, as numbers are; as becauſe in this Calculus known and unknown quantities are treated indifferently, and with the ſame facility; and laſtly, becauſe Analytical demonſtrations are general, and therefore applicable alike to all caſes; whereas in Arithmetick they are particular, and in every different caſe require a new determination.

Poſitive and negative quantities diſtinguiſhed.

2. Now of theſe quantities ſome are *poſitive*, or ſaid to be greater than nothing; others are leſs than nothing, and therefore are called *negative*. To explain this by an example. The goods in our own poſſeſſion may be called poſitive, but thoſe which we owe to others are negative, becauſe they muſt be ſubtracted from the poſitive, and therefore will diminiſh their ſum total. Wherefore, as the capitals in our poſſeſſion are poſitive, and are anſwerable for our debts; ſo the debts we owe will be negative quantities. In like manner, if a body or point in motion is directed towards a certain mark, and in it's paſſage deſcribes a ſpace, this ſpace may be called poſitive; but afterwards if it receives an oppoſite direction, it will indeed deſcribe a ſpace, but this ſpace will be negative in reſpect of the mark to which it ought to go. Wherefore, in Geometry, if a line drawn one way is aſſumed as poſitive, (for this is quite arbitrary,) a line drawn the contrary way will be negative.

The ſigns of poſitive and negative quantities, with other marks, explained.

3. Poſitive and negative quantities in Algebra are diſtinguiſhed by means of certain marks, or ſigns, which are prefixed to them. To poſitive quantities the ſign $+$, or *plus*, is prefixed: to negative quantities the ſign $-$, or *minus*. And when a quantity has no ſign prefixed, as when it ſtands alone, or is the firſt among others, it is then always ſuppoſed to be affected by the poſitive ſign. The ſign $\pm$, the contrary of which is $\mp$, is an ambiguous ſign, and ſignifies either *plus* or *minus*. So, for example, $\pm a$ inſinuates, that the quantity or number repreſented by a may be taken either affirmative (that is, poſitive) or negative. The mark $=$ ſignifies equality, and therefore $a = b$ informs us, that the two quantities expreſſed by a and b are equal to each other. So $a > b$ means, that a is greater than b. Alſo, $a < b$ tells us, that a is leſs than b. The equality of ratios, or the geometrical proportion of three or four terms, is thus expreſſed: $a . b :: b . c$, when there are three terms; that is, the ratio of a to b is equal to the ratio of b to c. Alſo, $a . b :: c . d$ means, that a is to b as c is to d. Laſtly, the ſign ∞ denotes infinite, and therefore $a = \infty$ ſignifies, that a is equal to infinite, or is an infinite quantity.

Quantities are divided into ſimple or compound.

4. A quantity is ſimple, incomplex, or of one term only, when it is expreſſed by one or more letters, but thoſe are not ſeparated or diſtinguiſhed from one another by the ſign either of addition or ſubtraction. Such are a, ab, aac, and the like. So, on the contrary, quantities are compound, or of ſeveral terms,

terms, when they are expreſſed by ſeveral letters, ſeparated from one another by the ſigns + or —. Such are $a + b$, $aa - ff + bb$, and the like. And therefore $a + b$ will be a quantity of two terms, or a binomial; $aa - ff + bb$ will be one of three terms, or a trinomial, &c.

Addition of Simple Quantities, being Integers.

5. Simple quantities are added to one another by writing one after another, prefixing to each it's proper ſign. As if we were to add a to b and c, the ſum would be repreſented by $a + b + c$. If we were to add a to $-b$, the ſum would be $a - b$. To add a to b to a to b, the ſum would be $a + b + a + b$. But here it muſt be obſerved, that $a + a$ is the ſame as $2a$, and $b + b$ is the ſame as $2b$; therefore the ſum will be $2a + 2b$. Therefore, to add the ſame quantities, or ſuch as are expreſſed by the ſame letters, it will ſuffice to prefix to the ſame letter ſuch a number as ſhall contain ſo many units, as are the times that the letter ſhould be repeated. Thus, the ſum of ac to ac to ac, that is, $ac + ac + ac$, will make $3ac$, and this number is called *the Numeral Co-efficient* of the quantity. And if the quantities to be added, being denominated by the ſame letter, ſhall have different co-efficients, thoſe co-efficients are to be added by the ordinary rules of Arithmetick. Thus the ſum of $2a$ and $5a$, together with b and $4b$, will be $7a + 5b$. And thus the ſum of a and $3b$, and $-2c$, and $7c$, and $5a$, will be $a + 3b - 2c + 7c + 5a$. But $a + 5a$ will make $6a$, and $-2c + 7c$ make $5c$. Therefore the ſum will be $6a + 3b + 5c$. Addition of ſimple quantities.

Subtraction of Simple Quantities, being Integers.

6. To ſubtract one quantity from another, the ſign muſt be changed of that quantity which is to be ſubtracted, and then with it's ſign ſo changed it muſt be wrote with the other. Thus to ſubtract b from a, we muſt write $a - b$; where it may be obſerved, that if a is a quantity greater than b, the remainder of the ſubtraction, or the difference, will be poſitive. But if b is greater than a, in that caſe the difference will be a negative quantity. To ſubtract aff from bbc, it will make $bbc - aff$. To ſubtract $2a$ from $5a$, it will make $5a - 2a$; but $5a$ leſſened by $2a$ make $3a$, ſo that the remainder will be $3a$. And to ſubtract $-b$ from a, it muſt be written $a + b$. Nor ſhould it ſeem ſtrange, that to ſubtract the negative quantity $-b$ it muſt become poſitive, that the remainder Subtraction of ſimple quantities.

 may

may be $a + b$; for as much as to ſubtract one quantity from another is the ſame thing as to find the difference between thoſe quantities. Now the difference between a and $-b$ is $a + b$, juſt in the ſame manner as the difference between a capital of 100 crowns and a debt of 50 is 150 crowns. For from having an hundred and having none, the difference is an hundred; and from having none to having a debt of fifty, the difference is fifty; therefore, from having an hundred to having a debt of fifty, the difference muſt be an hundred and fifty. Thus, for the ſame reaſon, to ſubtract b from $-a$, it muſt be written $-a - b$; and to ſubtract $-b$ from $-a$, it muſt be written $-a + b$.

Multiplication of Simple Quantities, being Integers.

Multiplication of ſimple quantities.

7. Simple quantities are multiplied by writing them one after another, without any ſign between, (unleſs ſometimes the mark $\times$,) and the reſulting quantity is called the *Product*, as alſo the quantities ſo multiplied are called the *Factors* or *Multipliers*. But as to the ſign which is to be prefixed to theſe products, the general rule is this; that if the quantities to be multiplied are both poſitive or both negative, then the poſitive ſign muſt always be prefixed to the product: but if one of thoſe quantities, whichever it is, is poſitive, and the other negative, then the negative ſign muſt always be prefixed to the product. The reaſon of this is, becauſe multiplication is nothing elſe but a geometrical proportion, of which the firſt term is unity, the ſecond and third terms are the two quantities which are to be multiplied together, and the fourth term is the product. And therefore being placed in a row, unity for the firſt term, one of the multipliers for the ſecond, and the other multiplier for the third; therefore, by the nature of geometrical proportion, the fourth muſt be ſuch a multiple of the third, as the ſecond is a multiple of the firſt. If the ſecond and third terms are poſitive, for example, if it is $1 . a :: b .$ to a fourth; the firſt term or unity being poſitive, the fourth muſt therefore be poſitive. But if the ſecond is negative, and the third poſitive, that is, if $1 . -a :: b .$ to a fourth; whereas this fourth muſt be ſuch a multiple of the third as the ſecond is of the firſt, and the ſecond being negative, therefore the fourth muſt be negative. Let the ſecond be poſitive and the third negative, that is, let it be $1 . a :: -b .$ to a fourth. Now, whereas this fourth muſt be ſuch a multiple of the third, as the ſecond is of the firſt, and the ſecond and firſt being poſitive and the third negative, the fourth cannot be otherwiſe than negative. Laſtly, let both the ſecond and third be negative, that is, let it be $1 . -a :: -b .$ to a fourth. Now the ſecond being here a negative multiple of the firſt, it follows that the fourth muſt be a negative multiple of the third. But the third is already negative, and therefore the fourth muſt be poſitive. Wherefore the product of a into b will be ab. That of a into $-b$ will be $-ab$. That of $-a$ into b

will alſo be $-ab$. That of $-a$ into $-b$ will be ab. That of a into b into c will be abc. That of a into $-b$ into c, will be $-abc$; becauſe a into $-b$ will be $-ab$, and $-ab$ into c will be $-abc$. And the product of $-a$ into $-b$ into c will be abc.

If the quantities to be multiplied ſhall have numeral co-efficients, thoſe co-efficients muſt be multiplied together by the common rules of numbers, and the product muſt be prefixed to that of the letters. Hence the product of $6a$ into $-8bc$ will be $-48abc$. And the product of $2a$ into $-2b$ into $-3c$ will be $12abc$. And the like of all others.

8. Now becauſe the product of a into a is aa, that of a into a into a, or of aa into a, is aaa, that of a into a into a into a, or of aaa into a, is $aaaa$, and ſo on ſucceſſively; to prevent the repetition of the ſame letter ſo often, it is uſual to write a^2 inſtead of aa, a^3 inſtead of aaa, a^4 inſtead of $aaaa$, and ſo of others. That is, we may write a little above the letter ſuch a number as ſhews the number of times the letter ought to be repeated, which number is called an *Index* or *Exponent*. We may write indifferently aa or a^2, but higher products or powers are more commonly expreſſed by their exponents. Notation of ſimple powers.

9. As the product of a number multiplied by itſelf is called the *Square* of that number, or it's ſecond power; ſo if this product is again multiplied by the ſame number, the new product is called the *Cube* or the third power of the ſame number. And the product of the cube by the ſame number is called the *Biquadrate* or fourth power of the ſame number; and ſo on. Thus the quantity a multiplied by a, or a^2, is called the Square of a, or the ſecond power of a; a^3 is it's cube or third power; a^4 it's fourth power, &c. Therefore $2a$ and a^2 will be very different from each other, the firſt being the ſum of a and a, or $a + a$, the other their product, or aa. The ſame is to be underſtood of $3a$ and a^3, of $4a$ and a^4; and ſo of others. Now as the product of $+$ into $+$, or of $-$ into $-$, is always poſitive; it proceeds from thence, that as well the ſquare of a as of $-a$ will be always poſitive, or aa. So on the other hand the cube of a, or a^3, will always be poſitive, but the cube of $-a$ will always be negative, or $-a^3$. For $-a$ into $-a$ makes aa, and aa into $-a$ makes $-a^3$. Thus the fourth power as well of $-a$ as of a will be poſitive. And in general, when the exponent of the power, to which we would raiſe the given quantity, is an even number, whether the quantity itſelf is poſitive or negative, that which reſults will always be poſitive. And when the exponent is an odd number, if the quantity is poſitive, the reſult will be poſitive; and if it be negative, the reſult or power will be negative. Names of the powers, and their diſtinction into poſitive and negative.

Diviſion.

Division of Simple Quantities, being Integers.

Division, what.

10. Divifion is an operation directly contrary to Multiplication; and what this compounds, that again refolves. Thus, becaufe ab is the product of a into b, therefore if we divide ab by a, we fhall have b for the quotient. And if we divide it by b, the quotient will be a. So dividing abc by c, we fhall have the quotient ab. And fo on. The quantity to be divided is called the *Dividend*, that by which the divifion is performed is called the *Divifor*, and that which refults from the divifion is called the *Quotient*, as in common Arithmetick. Therefore whenever in the dividend and the divifor the fame quantities are found, they may be taken out of both, or as it were cancelled, and what is left will give the quotient. Thus, if we are to divide aa by a, the quotient will be a. If we divide a^3 by a, the quotient will be aa. If we divide a^3b^3 by a^2b^2, the quotient will be ab. If the dividend and divifor fhall have numeral co-efficients befides, they are alfo to be divided by the common rules of Arithmetick, and the refulting numeral co-efficient muft be prefixed to the literal quotient. Thus, if we divide $3a^3b^3$ by $3b^3$, the quotient will be a^3. Dividing $56a^2b^3$ by $8ab$, the quotient will be $7ab^2$. And here it may be obferved, that whenever the quantity to be divided is the fame as the divifor, then the quotient will be unity; as dividing b by b, $7a^3$ by $7a^3$, and fuch like. The reafon of which is plain, becaufe to divide is to find how often the divifor is contained in the dividend, the anfwer of which queftion is the quotient.

When quotients are to be reprefented as fractions.

11. Wherefore, in the divifor and dividend, when no common quantities or letters are found, by means of which the divifion may be performed in the foregoing manner, the quotients will receive the form of fractions. Thus, to divide a by b, a^3 by bc, $5aabb$ by $2cc$, &c. the quotients are to be wrote thus: $\frac{a}{b}$, $\frac{a^3}{bc}$, $\frac{5aabb}{2cc}$, &c: that is, place the dividend above, and the divifor under it, with a little line between them; and it is to be underftood, that a ought to be divided by b, a^3 by bc, &c; and thefe are called *Fractions*, in which the quantity above the line is called the *Numerator*, and that below is the *Denominator*. Thus if any of the letters of the divifor, but not all, fhall be in common with the letters of the dividend, thofe that are common are to be taken away from each, and of thofe that remain a fraction is to be formed. Thus, if we were to divide a^3bb by $5abcc$, the quotient will be $\frac{a^3bb}{5abcc}$, or $\frac{a^2b}{5cc}$. And if we divide $10ab^3$ by $15bcc$, the quotient will be $\frac{2abb}{3cc}$. And fo of all others.

12. Now,

The ſign of the quotient, what.

12. Now, becauſe both the dividend and diviſor may be either poſitive or negative, it is neceſſary in every combination of caſes to fix a rule, for the ſign which is to be prefixed to the quotient. This rule is the ſame as that which ſerves for multiplication. That is to ſay, that if the dividend and the diviſor have both the ſame ſign, whether poſitive or negative, the quotient will be always poſitive. But if they have contrary ſigns, the quotient muſt be negative. The demonſtration depends on that of multiplication. For as multiplication is a proportion, of which the firſt term is unity, the ſecond and third are the two multipliers, and the fourth is the product; ſo diviſion is the ſame proportion, but inverted. Of this the firſt term is the dividend, the ſecond the diviſor, the third is the quotient, and the fourth is unity. Let it be required to divide $\pm ab$ by $\pm b$. Then the proportion will be $\pm ab \,.\, \pm b :: * a \,.\, 1$. Here I place the mark $*$ before the third term or quotient, as not yet knowing whether it ought to be poſitive or negative. Now, conſidering this proportion to be like that of multiplication, but the terms placed inverſely, it is known that when the ſecond term b is poſitive, the firſt term ab cannot be poſitive, unleſs the third term a is poſitive alſo; and the ſecond b being negative, the firſt ab cannot be negative, unleſs the third a be poſitive. Wherefore, in diviſion, when the two firſt terms, or the dividend and diviſor, are both poſitive or both negative, the third term, or quotient, muſt neceſſarily be poſitive. In like manner, in this proportion, the ſecond term b cannot be poſitive and the firſt ab negative, or the ſecond b negative and the firſt ab poſitive, unleſs the third a be negative. So that in diviſion, the dividend being poſitive and the diviſor negative, or on the contrary, the quotient of neceſſity muſt be negative.

Signs reciprocal in ſimple fractions.

13. For this reaſon it will be the ſame thing whether we write (for example) $\frac{a}{-b}$, or $\frac{-a}{b}$; becauſe if a poſitive is to be divided by b negative, or if a negative is to be divided by b poſitive, in both caſes the quotient muſt be negative. Thus it will be the ſame to write $\frac{-a}{-b}$, or $\frac{a}{b}$.

Extraction of the Roots of Simple Quantities, being Integers.

Roots of ſimple quantities extracted.

14. As quantities have their ſeveral powers, the ſquare, the cube, the biquadrate, the fourth power, &c. ſo among the roots of ſuch powers there is the ſquare-root or ſecond root, the cube-root or third root, the fourth root, &c. The denomination of roots is the ſame as that of the exponents of powers. Therefore the index or exponent of the ſquare-root is 2, of the cube-root is 3, &c. And to extract the root of a given quantity, we muſt find ſuch another quantity, as being multiplied into itſelf as many times, all but one, as are the

units

units in the index of the root, ſhall have for the product the quantity whoſe root is propoſed to be extracted. Thus a will be the ſquare-root of aa, the cube-root of a^3, the biquadratick-root of a^4, &c. In the ſame manner the ſquare-root of $aabb$ will be ab, of $16aabbcc$ will be $4abc$; the cube-root of $27a^3x^3$ will be $3ax$; and ſo of others.

Signs of roots. Impoſſible roots.

15. And ſince the product of *minus* into *minus* is always *plus*, as above; thence it follows that the ſquare-root of aa will be either a or $-a$, that is $\pm a$. It is not ſo with the cube-root, which will always be poſitive if the given cube is poſitive, and will be negative if this be negative; for the cube of a will be a^3, and the cube of $-a$ will be $-a^3$. But the fourth root will be either poſitive or negative. And to ſpeak in general, the roots whoſe index is an even number will always be either poſitive or negative; but thoſe whoſe index is an odd number will be poſitive if the power propoſed be poſitive, and negative if that be negative. And hence it is, from the ſame property of the ſigns in multiplication, that no poſitive or negative quantity can ever produce a negative power having an even exponent. So that it is impoſſible to find the root of a negative power with an even exponent. Such roots as theſe, of a negative quantity with an even index, are therefore called *Impoſſible* or *Imaginary*. Thus the ſquare-root of $-aa$ will be imaginary, as alſo the fourth root of $-a^4$, the ſixth root of $-a^6$, &c. But ſuch as theſe will be true and real roots, the cube-root of $-a^3$, the fifth root of $-a^5$, &c.

Roots extracted of imperfect powers.

16. But for the generality the quantities propoſed, of which we are to extract the roots, will not be true ſquares, cubes, or other powers, which are produced by the multiplication of rational quantities into themſelves, but will be the products of another kind; as ab, abc, &c: in which caſe we make uſe of the mark $\surd$, called the *Radical Sign* or *Vinculum*. Hence $\sqrt[2]{ab}$, or ſimply $\sqrt{ab}$, denotes the ſquare-root of ab; $\sqrt[3]{abc}$ denotes the cube-root of abc. And thus $\sqrt[4]{}$ ſtands for the fourth or biquadratick-root, $\sqrt[5]{}$ ſtands for the fifth root, &c. And ſuch quantities as theſe, affected by a radical ſign or vinculum, are called *Surds*, or *Irrational Quantities*.

Addition of Compound Quantities, being Integers.

Compound quantities added.

17. By the addition or ſubtraction of ſimple quantities, compound quantities are produced. Therefore, to add theſe together, it is ſufficient to write them one after another with their proper ſigns. So to add $a + b$ to $c - d$, we may write $a + b + c - d$. To add $2aa - xx$ to $3cc + 2yy$, the ſum will be $2aa - xx + 3cc + 2yy$. To add $aa - xx$ to $bb + xx + yy$, we ſhall have $aa - xx + bb + xx + yy$; but here it is to be obſerved, that $-xx$ and $+xx$ remove

remove or destroy each other, and therefore may both be cancelled or omitted, and then the sum will be $aa + bb + yy$. To add $2aa - 5bb$ to $aa + 2bb + yy$, the sum will be $2aa - 5bb + aa + 2bb + yy$; but here $2aa + aa$ make $3aa$, and $-5bb + 2bb$ make $-3bb$, and therefore the sum will be $3aa - 3bb + yy$.

Subtraction of Compound Quantities, being Integers.

18. The signs must be changed of that quantity which is to be subtracted, and then with the signs so changed it is to be wrote after that, from which the subtraction is to be made. Thus to subtract $c - d$ from $a + b$, we must write them thus, $a + b - c + d$; and the reason is plain. For if we were to subtract only the quantity c, we should write $a + b - c$. And now having subtracted too much, (for we ought to have subtracted only $c - d$, or the difference between c and d,) having subtracted, I say, more than we ought by the quantity d, to make amends we must add d, and so write the remainder $a + b - c + d$. The same is to be done for quantities more compounded. To subtract $a + 3b$ from $3a + 2b$, it will be wrote $3a + 2b - a - 3b$; but by a reduction of similar terms, because $3a - a$ is $2a$, and $2b - 3b$ is $-b$, the remainder will become $2a - b$. To subtract $3ab - 2bc + 2cd$ from $5ab - 4bc + 2cd$, after a proper reduction the remainder will be $2ab - 2bc$. Compound quantities subtracted.

Multiplication of Compound Quantities, being Integers.

19. The rule for the multiplication of simple quantities being understood, that for compound quantities will be very easy. Let one of the factors be wrote under the other, as is usual in the vulgar Arithmetick, and then all the terms of the multiplicand must be multiplied successively by every term of the multiplier, according to the rules already given for the multiplication of simple quantities; and what results, after the usual reduction of similar terms, will be the product required. Thus if we are to multiply $a + b - c$ by x, let them be wrote as in the margin. Then let every term of the multiplicand, placed above, be multiplied by the multiplier placed under it, and the product will be $ax + bx - cx$, as by the operation. Thus if we were to multiply Compound quantities multiplied.

$$\begin{array}{r} a + b - c \\ x \\ \hline ax + bx - cx \end{array}$$

$$\begin{array}{l} 2a + 3b - c \\ \quad\;\; 3x - 2y \\ \hline 6ax + 9bx - 3cx \\ \quad - 4ay - 6by + 2cy \end{array}$$

multiply $2a + 3b - c$ by $3x - 2y$, let them be placed as in the margin. Then multiply all the terms above by $3x$, and do the same by the other term $- 2y$, and so if there were more terms in the multiplier. The product will be as is here to be seen. It is no matter whether the operation begins from the right hand or from the left, in regard to either of the factors; or which of them is wrote above, and the other below; or in what order the terms are placed. Suppose we were to multiply $aa + xx$ by $aa - xx$; proceed as in the margin, where because $aaxx$ and $- aaxx$ destroy each other, the product will be reduced to $a^4 - x^4$.

$$\begin{array}{l} aa + xx \\ aa - xx \\ \hline a^4 + a^2x^2 \\ \quad - a^2x^2 - x^4 \end{array}$$

In long multiplications, in order to reduce similar terms with greater ease, it will be convenient to write those similar terms, which will arise from the multiplication, one under another as in the foregoing and following example. Let it be proposed to multiply $4a^3 + 3a^2b - 2ab^2 + b^3$ by $a^2 - 5ab + 6b^2$. The work will stand as in the margin.

$$\begin{array}{l} 4a^3 + 3a^2b - 2ab^2 + b^3 \\ \qquad\quad a^2 - 5ab + 6b^2 \\ \hline 4a^5 + 3a^4b - 2a^3b^2 + a^2b^3 \\ \quad - 20a^4b - 15a^3b^2 + 10a^2b^3 - 5ab^4 \\ \qquad\quad + 24a^3b^2 + 18a^2b^3 - 12ab^4 + 6b^5. \end{array}$$

Here it is easily perceived, that $+ 3a^4b - 20a^4b$ make $- 17a^4b$. And that $- 2a^3b^2 - 15a^3b^2 + 24a^3b^2$ make $7a^3b^2$. And that $+ a^2b^3 + 10a^2b^3 + 18a^2b^3$ make $29a^2b^3$. And that $- 5ab^4 - 12ab^4$ make $- 17ab^4$. Therefore the product is $4a^5 - 17a^4b + 7a^3b^2 + 29a^2b^3 - 17ab^4 + 6b^5$.

Multiplication how insinuated.

20. Sometimes it will be unnecessary actually to perform the multiplication in this manner, but it may be sufficient to insinuate it only, which is commonly done by inserting this mark $\times$, and drawing a line or *vinculum* over each of the multipliers, extended over all the terms which enter the multiplication. Thus $\overline{aa + xx} \times \overline{aa - xx}$ denotes the product of $aa + xx$ by $aa - xx$. But in the quantity $\overline{aa + xx} \times \overline{aa - xx} \pm a^4$, the term $\pm a^4$, not being included in the vinculum, is not intended to be comprehended in the multiplication. So that being wrote in this manner it must be understood, that to or from the product of $aa + xx$ into $aa - xx$, must be further added or subtracted the term a^4.

Powers of compound quantities how insinuated: how actually formed.

21. After the same manner that in simple quantities the product of a into a is called the square of a, the product of aa into a is called the cube of a, the product of a^3 into a is called the biquadrate of a, &c. So in compound quantities the product of $a + b$ (for example) into $a + b$, or $\overline{a + b} \times \overline{a + b}$, is called the square of $a + b$, which is wrote thus, $\overline{a + b}|^2$, when we would not actually form it by multiplication. In the same manner $\overline{a + b}|^2 \times \overline{a + b}$ will be the cube, which may be wrote thus, $\overline{a + b}|^3$; and $\overline{a + b}|^3 \times \overline{a + b}$, or

$\overline{a + b}|^2$

$\overline{a+b}|^2 \times \overline{a+b}|^2$, or $\overline{a+b}|^4$ will be the fourth power of $a+b$. And this is to be underſtood of quantities of any number of terms.

Actually to form theſe powers, the quantity given muſt be multiplied into itſelf, and the product by the ſame quantity ſucceſſively, as many times, ſave one, as the exponent of the power required contains unity. But for the ſecond power, or the ſquare, the operation may be thus abbreviated. If the quantity given is a binomial, or conſiſts only of two terms, ſuppoſe $a \pm b$, write down the ſquare of the firſt term, then the two rectangles, or twice the product of the firſt term by the ſecond, with ſuch a ſign as the rule of multiplication requires; and laſtly the ſquare of the ſecond term muſt be added. Thus $\overline{a+b}|^2$ will be $aa + 2ab + bb$; and $\overline{a-b}|^2$ will be $aa - 2ab + bb$. Alſo $\overline{-a-b}|^2$ will be $aa + 2ab + bb$. If the quantity given is a trinomial, or conſiſts of three terms; beſides the ſquare of the two firſt terms found as before, muſt be wrote two rectangles of the firſt into the third, and alſo of the ſecond into the third, (taking care that theſe rectangles may have their proper ſigns, according to the rules of multiplication,) and laſtly the ſquare of the third term. Thus $\overline{a+b-c}|^2$ will be $aa + 2ab + bb - 2ac - 2bc + cc$. If the quantity is a quadrinomial, or of four terms, there muſt be wrote beſides, twice the rectangles of the three firſt terms into the fourth, and alſo the ſquare of the fourth term. And ſo on to other multinomials.

22. But as to all binomial quantities, the following general canon will be of good uſe, not only to raiſe it to the ſquare, but to any power denoted by m, where m ſtands for any number whatever. Therefore let $p+q$ be to be raiſed to the power m; this power will be $p^m + mp^{m-1}q + m \times \frac{m-1}{2}p^{m-2}q^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3}p^{m-3}q^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}p^{m-4}q^4$, &c.; which ſeries of terms may be continued as far as we pleaſe, obſerving the ſame law. Powers raiſed by the *Binomial Theorem* of Sir I. N.

From hence let us derive the ſquare of $p+q$. In this caſe m will be 2, and therefore in this canon, by ſubſtituting 2 inſtead of m, the firſt term will be p^2; the ſecond $2p^{2-1}q$, that is $2pq$; the third will be $2 \times \frac{2-1}{2}p^{2-2}q^2$, that is q^2. (Here we do not admit the quantity p, becauſe being raiſed to no power, it is equal to unity, as will be ſhown afterwards. And the fourth term will be $2 \times \frac{2-1}{2} \times \frac{2-2}{3}p^{2-3}q^3$. But $2-2$ in the co-efficient is equal to nothing, and therefore this term being multiplied by nothing will be nothing, or will vaniſh. And thus ſince all the following terms are multiplied by nothing, they will all vaniſh, and the canon will terminate after three terms. So then the ſquare required will be $pp + 2pq + qq$.

If we would have the cube or third power of $p + q$, then $m = 3$; whence the fifth term of the canon, and all the following ones, will be equal to nothing. So that the power required, by ſubſtituting 3 inſtead of m, will be $p^3 + 3p^2q + 3pq^2 + q^3$. If the quantity to be raiſed is $p - q$, it will be ſufficient to place the ſign *minus* before all the terms, in which the index of q is an odd number.

The foregoing canon will not only ſerve for the binomial $p \pm q$, but for any other whatever. So that if we would have the third power of $2ax - xx$, we muſt ſuppoſe $p = 2ax$, and $q = -xx$, as alſo $m = 3$. Then in the canon, inſtead of p and the powers of p, we muſt ſubſtitute $2ax$ and it's powers; which muſt alſo be done by putting $-xx$ inſtead of q and it's correſponding powers. Then inſtead of m put 3, and the cube will be $8a^3x^3 - 12aax^4 + 6ax^5 - x^6$.

It may likewiſe ſerve for any polynome, or for any quantity conſiſting of more terms than two. Let there be a trinomial $a + b - c$ to be raiſed to the third power, and then it will be $m = 3$. If we make $p = a$ and $q = b - c$, and ſubſtitute a and it's powers inſtead of p and it's powers, and alſo $b - c$ and it's powers inſtead of q and it's powers; the cube will be $a^3 + 3aa \times \overline{b - c} + 3a \times \overline{b - c}|^2 + \overline{b - c}|^3$; that is, $a^3 + 3a^2b - 3a^2c + 3ab^2 - 6abc + 3ac^2 + b^3 - 3b^2c + 3bc^2 - c^3$.

Diviſion of Compound Quantities, being Integers.

Compound quantities divided.

23. There may be three different caſes, or combinations, in the diviſion of compound quantities; the firſt is, when the quantity to be divided is compound, and the diviſor is ſimple; the ſecond is on the contrary, when the diviſor is compound, and the dividend ſimple; the third is when they are both compound quantities. As to the two firſt caſes, it will ſuffice to make uſe of the rule for ſimple quantities. In the firſt caſe every term of the quantity propoſed is divided by the diviſor, and there will ariſe either integers or fractions, as follows from the nature of diviſion of ſimple quantities. Thus if we are to divide $aa + ab - ac$ by a, we ſhall have for the quotient $a + b - c$. If we are to divide $4ab - 6bc + xx$ by $2b$, we ſhall have $2a - 3c + \frac{xx}{2b}$. If we are to divide $4ab - cc + 3xx$ by $3c$, we ſhall have $\frac{4ab - cc + 3xx}{3c}$, or elſe $\frac{4ab}{3c} - \frac{c}{3} + \frac{xx}{c}$. In the ſecond caſe the diviſor is wrote under the dividend, as is uſual in fractions; and if in every term of the numerator and denominator

nator there ſhall be any common quantity, it may be cancelled; then what remains will always be a fraction. Thus dividing $3a^3b$ by $aa - ax + ab$, the quotient will be $\frac{3aab}{a-x+b}$. And if we divide $6a^4$ by $2aa - 2ax + 2xx$, the quotient will be $\frac{3a^4}{aa - ax + xx}$.

24. In the third caſe it is neceſſary, firſt to put the terms of the dividend in order, and likewiſe of the diviſor, in reſpect to ſome certain letter which ſhall be thought the moſt proper for that purpoſe. This is done by writing that for the firſt term of the dividend, and alſo of the diviſor, in which that letter is found of the higheſt power, or of moſt dimenſions. Then making that the ſecond term, in which that letter is of the next greateſt power. And ſo ſucceſſively till we come to thoſe terms, which are not affected by that letter at all, which therefore muſt be made the laſt. Thus the quantity $a^3 + 2a^2c - a^2b - 3abc + bbc$ will be ordered in reſpect of the letter a, and alſo the diviſor $a - b$. If we would diſpoſe this in order, in reſpect of the letter b, it muſt be done thus; $b^2c - 3abc - a^2b + a^3 + 2a^2c$; and the diviſor thus, $-b + a$. Proceſs of diviſion.

This ſuppoſed, the diviſion muſt be performed after this manner. The firſt term of the dividend muſt be divided by the firſt term of the diviſor, and the quotient muſt be written on one ſide. By this quotient the whole diviſor muſt be multiplied, and the product ſubtracted from the dividend. When the ſubtraction is made, and the terms reduced, in the ſame manner the firſt term of the remainder muſt be divided by the firſt term of the diviſor, and this term of the quotient muſt be wrote after the other, with ſuch ſign as it ought to have. Then the whole diviſor muſt be multiplied by this ſecond quotient, and the product ſubtracted from the dividend, that is from the firſt remainder. And proceeding in this manner, the calculation muſt be repeated, till at laſt there is no remainder. Then the ſum of all theſe quotients, thus found by parts, will be the whole quotient of the diviſion.

Let it be required to divide $a^3 + 2a^2c - a^2b - 3abc + b^2c$ by $a - b$. Let the quantity to be divided be wrote at A, the diviſor at B. Now dividing a^3 by a, the quotient will be a^2, which is written at D. Then finding the product of the quotient into the diviſor, and ſubtracting it from the dividend, there will be left the firſt remainder, as at M. Then dividing the firſt term $2aac$ in this remainder M by the ſaid firſt term of the diviſor a, and writing the quotient $2ac$ after the other at D, we muſt ſubtract the product of $2ac$ into the diviſor B, and we ſhall have the ſecond remainder N. Divide the firſt term $- abc$ of this ſecond remainder by the ſame term a of the diviſor, and write the quotient $- bc$ at D after the other. The product of $- bc$ into the diviſor muſt be ſubtracted from the ſecond remainder, and nothing will now remain. Therefore the compleat quotient will be $aa + 2ac - bc$.

A.	$a^3 + 2a^2c - a^2b - 3abc + b^2c$	B. $a - b$
M.	$2a^2c - 3abc + b^2c$	D. $aa + 2ac - bc$.
N.	$- abc + b^2c$	

Let $a^3 - 3a^2b + 3ab^2 - b^3$ be to be divided by $a - b$. Let the dividend be wrote at A, and the divifor at B. Let the firft term a^3 be divided by a, and the quotient aa be wrote at D. Then finding the product of the quotient into the divifor, and fubtracting it from the dividend, there will be left the firft remainder M. Let the firft term of this remainder, that is $- 2a^2b$, be di-divided by the fame firft term of the divifor a, and let the quotient $- 2ab$ be wrote after the other at D. Then let the product of $- 2ab$ into the divifor be fubtracted from the firft remainder M, and we fhall have the fecond remainder N. If we divide the firft term ab^2 of this fecond remainder by the fame firft term of the divifor a, the quotient bb muft be wrote at D after the other. Then let the product of bb into the divifor B be fubtracted from the fecond remainder N, and nothing will remain; fo that the whole quotient will be $aa - 2ab + bb$.

A.	$a^3 - 3a^2b + 3ab^2 - b^3$	B. $a - b$
M.	$- 2a^2b + 3ab^2 - b^3$	D. $aa - 2ab + bb$
N.	$+ ab^2 - b^3$	

Another Example.

A.	$2aa + 5ab + 2bb - ac - 2bc$	B. $a + 2b$
M.	$+ ab + 2bb - ac - 2bc$	D. $2a + b - c$
N.	$- ac - 2bc$	

Another.

A.	$9d^4 + 12d^3e - 4de^3 - e^4$	B. $3d^2 - e^2$
M.	$12d^3e + 3d^2e^2 - 4de^3 - e^4$	D. $3d^2 + 4de + e^2$
N.	$3d^2e^2 - e^4$	

Another.

A.	$4a^2 + 4ab - 2ac + b^2 - c^2$	B. $2a + b$
M.	$2ab - 2ac + b^2 - c^2$	D. $2a + b - c$
N.	$- 2ac - c^2$	
O.	$bc - c^2$	

Now here it is to be obferved, that the laft remainder at O is not divifible by $2a$, and confequently the operation cannot proceed, but it muft remain as a fraction $\frac{bc - cc}{2a+b}$. That is to fay, that the quantity propofed is not entirely

divisible by $2a + b$, but only in part, and therefore the quotient will be partly an integer, and partly a fraction, as $2a + b - c + \frac{bc - cc}{2a+b}$. Or the whole may be wrote as a fraction thus, $\frac{4aa + 4ab - 2ac + bb - cc}{2a + b}$.

Extraction of the Roots of Compound Quantities, being Integers.

25. As in simple quantities, so in compound; the root of any quantity is that, which being multiplied into itself, if once produces the given square, if twice produces the given cube, and so on. Roots how to be extracted; particularly the square-root.

The manner of extracting the square-root in compound quantities is as follows: It being first understood, that the terms must be disposed in order, according to some one of it's letters, agreeably to the caution before given, § 24.

Let the given quantity be $a^2 + 2ab + b^2$, whose root is to be extracted, and let it be wrote down as at A. Extract the square-root of the first term a^2, which will be a, and let it be wrote as at B. The square of this, or a^2, must be subtracted from the quantity proposed, A, and the remainder wrote down at D. Then the quantity a, wrote down at B, must be doubled, and wrote as at M, which will be $2a$. By this quantity $2a$ the first term at D must be divided, and the quotient b wrote at B. Then the divisor $2a$ must be multiplied by the quotient b, and the product subtracted from the quantity D; and moreover the square of b must be subtracted from the same; and as there is no remainder, the root required will be $a + b$.

A.	$a^2 + 2ab + b^2$	B.	$a + b$
D.	$2ab + b^2$	M.	$2a$

Let the quantity given be $a^4 + 6a^3b + 5a^2b^2 - 12ab^3 + 4b^4$; let it be wrote at A, and let the square root of the first term be extracted, which is a^2, and let this root be wrote at B. Let the square of a^2 be subtracted from the quantity A, and there will remain the quantity D. Let a^2 be doubled and wrote at M, and by this double, that is by $2a^2$, let the first term be divided of the first remainder D, and the quotient $3ab$ be wrote at B. Then subtracting the product of $3ab$ into the divisor $2aa$, as also the square of $3ab$, from the first remainder D, there will be left the second remainder H. Let the whole quantity B be doubled, and wrote at G. By it's first term let the first term of H be divided, and the quotient $- 2b^2$ be wrote at B. Then subtracting the product of the quotient into the divisor G, and also the square of the same quotient,

quotient, from the quantity H; and, as there is no remainder, the quantity written at B, that is, $aa + 3ab - 2bb$, will be the root required.

A.	$a^4 + 6a^3b + 5a^2b^2 - 12ab^3 + 4b^4$	B.	$a^2 + 3ab - 2b^2$
D.	$6a^3b + 5a^2b^2 - 12ab^3 + 4b^4$	M.	$2a^2$
H.	$-4a^2b^2 - 12ab^3 + 4b^4$	G.	$2a^2 + 6ab$

The Operation of another Example.

A.	$y^4 + 4ay^3 - 8a^3y + 4a^4$	B.	$y^2 + 2ay - 2a^2$
D.	$4ay^3 - 8a^3y + 4a^4$	M.	$2y^2$
H.	$-4a^2y^2 - 8a^3y + 4a^4$	G.	$2y^2 + 4ay$

Another Example.

A.	$16a^4 - 24a^2x^2 - 16a^2b^2 + 12b^2x^2 + 9x^4$	B.	$4a^2 - 3x^2 - 2b^2$
D.	$-24a^2x^2 - 16a^2b^2 + 12b^2x^2 + 9x^4$	M.	$8a^2$
H.	$-16a^2b^2 + 12b^2x^2$	G.	$8a^2 - 6x^2$
K.	$-4b^4$		

In this laſt operation there is a remainder of $-4b^4$, which cannot be divided by $8a^2$, as the method requires, which in this caſe cannot take place. That is to ſay, that the ſquare-root of the propoſed quantity cannot be actually extracted, and therefore we muſt make uſe of the radical ſign, as above at § 16; which expedient muſt alſo be applied in other extractions, as the cube-root, the biquadratick-root, &c. Thus $\sqrt{aa + bb}$ repreſents the ſquare-root of $aa + bb$; and $\sqrt[3]{aab - abb}$ will ſtand for the cubic root of $aab - abb$; and the like for other roots.

The cube-root extracted.

26. As to the cube-root, let it be required to extract the root of the quantity $a^3 + 3a^2b + 3ab^2 + b^3$, as is written below at A. Extract the cube-root of the firſt term a^3, which is a, and is written at B. Let the cube of this, or a^3, be ſubtracted from the given quantity A, and let the remainder be written at D. Then take the triple of the ſquare of a, which is $3aa$, and let it be wrote at M, by which divide the firſt term of the remainder D, and let the quotient b be wrote at B. By this multiply the diviſor $3aa$, and the product, together with the triple of the ſquare of b into a, and the cube of b, muſt be ſubtracted from the remainder D. And as nothing remains, $a + b$ will be the root required.

A.	$a^3 + 3a^2b + 3ab^2 + b^3$	B.	$a + b$
D.	$3a^2b + 3ab^2 + b^3$	M.	$3aa$

Let it be required to extract the cube-root of the quantity $z^6 + 6bz^5 - 40b^3z^3 + 96b^5z - 64b^6$.

Extract

Extract the root of the first term z^6, which is z^2, and let it be wrote at B. Let the cube of B be subtracted from the proposed quantity A, and let the remainder be wrote at D. Take the triple of the square of B, and write it at M, and by that divide the first term of the remainder D, and write the quotient $2bz$ at B. Then subtract the product of $2bz$ into the quantity M, and moreover the triple of the square of $2bz$ multiplied into zz, with the cube of $2bz$, from the remainder D, and write the remainder at H. Then find the triple of the square of B, which write in G, and by the first term divide the first term of the remainder H, and write the quotient $-4bb$ in B. Then multiply this quotient by the quantity G, and the product, together with the triple of the square of $-4bb$ into $zz + 2bz$, and the cube of $-4bb$ must be subtracted from the quantity H, and nothing will remain. Whence the cube-root of the quantity proposed will be the whole quantity B, that is, $zz + 2bz - 4bb$.

A. $z^6 + 6bz^5 - 40b^3z^3 + 96b^5z - 64b^6$ B. $z^2 + 2bz - 4b^2$
D. $6bz^5 - 40b^3z^3 + 96b^5z - 64b^6$ M. $3z^4$
H. $-12b^2z^4 - 48b^3z^3 + 96b^5z - 64b^6$ G. $3z^4 + 12bz^3 + 12b^2z^2$

After the same manner is extracted the cube-root of the following quantity.

A. $27y^6 - 54cy^5 + 144c^2y^4 - 152c^3y^3 + 192c^4y^2 - 96c^5y + 64c^6$
D. $- 54cy^5 + 144c^2y^4 - 152c^3y^3 + 192c^4y^2 - 96c^5y + 64c^6$
H. $108c^2y^4 - 144c^3y^3 + 192c^4y^2 - 96c^5y + 64c^6$

B. $3y^2 - 2cy + 4c^2$
M. $27y^4$
G. $27y^4 - 36cy^3 + 12c^2y^2$

27. For the fourth root. Let the quantity proposed be $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, of which we would extract the biquadratick or fourth root. Let it be wrote at A, and extract the fourth root of the first term, which is a, and write it at B. Subtract the fourth power of B from the quantity A, and write the remainder at D. Then find the quadruple of the cube of a, and write it at M. By this must be divided the first term of the quantity D, and the quotient b must be wrote at B. From the quantity D must be subtracted the product of the quotient b into the divisor $4a^3$, and moreover the sextuple of the square of b into the square of a, and the product of the quadruple of the cube of b into the quantity a, and lastly the biquadrate of b. And as there is no remainder, the root required will be $a + b$.

The fourth root extracted.

A. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ B. $a + b$
D. $4a^3b + 6a^2b^2 + 4ab^3 + b^4$ M. $4a^3$

The fifth and higher roots extracted.

28. As to the fifth root; in order to difcover in what manner the operations proceed, which are to be made in the extraction, it will be fufficient to form the fifth power of a binomial, fuppofe of $a+b$, which will give a rule here; as the fecond, third, and fourth powers of the fame binomial have fupplied us with rules for the extraction of the fecond, third, and fourth roots. The like obtains in the fixth, feventh, and other roots.

Of Fractions, Simple and Compound.

Notation of fractions.

29. We have feen before, how fractions or broken numbers arife from the divifion of quantities. Therefore a fraction infinuates a divifion that is to be made, of the numerator by the denominator. Whence it proceeds, that if the numerator is the fame as the denominator, as $\frac{a}{a}$, or $\frac{aa-bb}{aa-bb}$, and fuch like, thofe fractions can fignify nothing elfe but unity; becaufe in fact, if we divide a by a, or $aa-bb$ by $aa-bb$, the quotient will be unity. And becaufe multiplication is an operation contrary to divifion, it is plain, that any integer whatever may be reduced to a fraction with what denominator we pleafe, if it is multiplied by the quantity which is to be the denominator, and then divided by it again. Thus to reduce the integer a to a fraction with the denominotor b, we muft write $\frac{ab}{b}$. To reduce $a-b$ to a fraction with the denominator d, we muft write $\frac{ad-bd}{d}$. To reduce $a+b$ to a fraction whofe denominator fhall be $c-d$, we muft write $\frac{\overline{a+b}\times\overline{c-d}}{c-d}$, or $\frac{ac+bc-ad-bd}{c-d}$.

Reduction of Fractions to more fimple Expreffions.

How fractions are to be reduced.

30. When fractions have the fame letter or letters in every term of the numerator and denominator, it will be fufficient to expunge the common letters in both; having regard to their powers, as is faid in Divifion, at § 10. Thus $\frac{a^3b^2}{ac}$ will become $\frac{a^2b^2}{c}$; $\frac{ab^3}{abc}$ will be $\frac{bb}{c}$; $\frac{a^3b-x^3b}{ab-bb}$ will be $\frac{a^3-x^3}{a-b}$. But though there are not the fame letters in both the numerator and denominator, yet if each of them is multiplied by the fame compound quantity, they may be divided

divided by it again, and confequently the fraction may be reduced. Thus $\frac{aac - aad}{cd - dd}$, that is $\frac{aa \times \overline{c-d}}{d \times \overline{c-d}}$, will be reduced to $\frac{aa}{d}$. So $\frac{\overline{aa+2ab}|^2}{aab+2abb}$, that is $\frac{\overline{aa+2ab} \times \overline{aa+2ab}}{b \times \overline{aa+2ab}}$, will be reduced to $\frac{aa+2ab}{b}$. So $\frac{aac - aad - acd + add}{cd - dd}$, or $\frac{\overline{aa-ad} \times \overline{c-d}}{d \times \overline{c-d}}$, will be reduced to $\frac{aa-ad}{d}$.

Therefore in general, as often as the fraction is fuch, that it's numerator and denominator are both divifible by one and the fame quantity, which in this cafe is called their common divifor, by actually dividing both, the two quotients will give the fraction reduced. But it muft be obferved, that, if that common divifor is not the greateft that can be, the fraction indeed will be reduced, but not to the fimpleft expreffion. Thus the fraction $\frac{a^3 - abb}{aac + abc}$, that is $\frac{a \times \overline{a+b} \times \overline{a-b}}{a \times c \times \overline{a+b}}$, may be divided, both as to it's numerator and denominatoı, by a, by $a + b$, and by $aa + ab$, the greateft of which divifors is $aa + ab$. And as the fraction fhould be reduced to it's leaft terms, we muft divide it by $aa + ab$, and the quotient or fraction reduced will be $\frac{a-b}{c}$. But very often it will be difficult to know if there is a common divifor, and what it is; and therefore we fhall give a rule to find it, at § 36. afterwards. At prefent we fhall omit it, that we may not too much difcourage young learners, as yet not fufficiently confirmed, and fhall proceed to other operations; making ufe of fractions that are any how reduced to lower and fimpler expreffions.

Reduction of Fractions to a Common Denominatcr.

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31. If two fractions are given, let the numerator of the firft be multiplied by the denominator of the fecond, and the numerator of the fecond be multiplied by the denominator of the firft, and each product be divided by the product of the two denominators. Thus $\frac{a}{b} + \frac{x}{y}$ will be $\frac{ay + bx}{by}$; and $\frac{a^3}{y^2} - \frac{2x^2}{3b}$ will be $\frac{3a^3b - 2x^2y^2}{3by^2}$. Alfo $\frac{aa + xx}{m+n} - \frac{aa}{m}$ will be $\frac{ma^2 + mx^2 - ma^2 - na^2}{mm + mn}$, that is $\frac{mxx - naa}{mm + mn}$. But here we muft take notice, that as often as the two denominators of the fractions have a greateft common divifor, in this cafe the multiplication

Fractions reduced to a common denominator.

plication of the numerators into that common divifor is fuperfluous, and alfo of thofe common divifors into each other, for forming a new denominator; for then it may be neceffary to reduce the fractions to more fimple expreffions. Wherefore the faid numerators fhould be multiplied, not by the denominators, but by the quotients which will refult by dividing the faid denominators by their common divifors: and the denominator will be the product of thofe quotients, and of the faid common divifor. For example, let there be given $\frac{a^3}{mn} + \frac{abb}{mx}$. Being reduced as ufual to a common denominator, it will be $\frac{ma^3x + mnabb}{mmnx}$; that is $\frac{a^3x + nabb}{mnx}$. Therefore it was needlefs to multiply the numerators by m, the common divifor of the denominators, as it was fuperfluous to multiply the denominators together. It was fufficient to multiply a^3 into x, and abb into n, to form the numerators, and to multiply m into n into x, to form the common denominator. Thus to reduce to a common denominator the fractions $\frac{a^3 - b^3}{\overline{a+b}|^2} - \frac{aa}{a+b}$, it will be enough to multiply $-\frac{aa}{a+b}$ into $a + b$, and it will be $\frac{a^3 - b^3 - a^3 - aab}{\overline{a+b}|^2}$, that is $\frac{-b^3 - aab}{\overline{a+b}|^2}$. In like manner to reduce to a common denominator the fractions $\frac{b^4}{a^2c - a^2d} + \frac{a^3 + b^3}{cd - dd}$; becaufe $c - d$ is a common divifor of both the denominators, it will fuffice to multiply b^4 by d, and $a^3 + b^2$ by a^2, as to the numerators; and to multiply a^2 into d into $c - d$, as to the denominator, and therefore it will be $\frac{b^4d + a^5 + a^2b^3}{a^2cd - a^2d^2}$.

If three fractions are to be reduced to a common denominator, let the two firft be reduced, then that which refults from thefe with the third; and fo on fucceffively if there are more. So to reduce thefe to a common denominator, $\frac{a}{b} + \frac{c}{d} - \frac{m}{n}$, let the two firft be reduced, and we fhall have $\frac{ad + bc}{bd}$. Let this be reduced with the third, and we fhall have $\frac{adn + bcn - bdm}{bdn}$. This may alfo be done in refpect to integers; for whereas any integer may be confidered as a fraction, having unity for it's denominator, we may proceed after the fame manner as before. Thus $2aa + \frac{3x^4 - 2y^4}{3x^2 - 8ax}$, that is $\frac{2aa}{1} + \frac{3x^4 - 2y^4}{3x^2 - 8ax}$, will be $\frac{6a^2x^2 - 16a^3x + 3x^4 - 2y^4}{3xx - 8ax}$.

Addition and Subtraction of Fractions.

32. Fractions are added by writing them one after another with the ſame ſigns. And on the contrary they are ſubtracted by changing the ſigns of the quantities to be ſubtracted. And the ſame things muſt be done, if there are integers with the fractions. Thus to add $\frac{aa}{c}$ to $\frac{bb}{c}$, they are wrote $\frac{aa+bb}{c}$. To add $\frac{aa}{c}$ to $\frac{xx}{m}-y$, it muſt be wrote $\frac{aa}{c}+\frac{xx}{m}-y$; which afterwards (if we pleaſe) may be reduced to a common denominator, and then it will be $\frac{aam+cxx-cmy}{cm}$. To add $\frac{aab^4}{a^4-2a^2b^2+b^4}$ to $\frac{aabb}{aa-bb}$, the ſum will be $\frac{aab^4}{a^4-2a^2b^2+b^4}+\frac{a^2b^2}{aa-bb}$, which if we would further reduce to a common denominator, we may obſerve, that the denominator of the firſt is the ſquare of $aa-bb$; therefore the two denominators have a greateſt common diviſor $aa-bb$, by which being divided, the quotients will be $aa-bb$ in the firſt, and unity in the ſecond. Wherefore it will be enough to multiply the numerator of the ſecond fraction by $aa-bb$, and to divide the whole by $a^4-2a^2b^2+b^4$, and the ſum required will be $\frac{a^2b^4+a^4bb-a^2b^4}{a^4-2aabb+b^4}$, that is $\frac{a^4bb}{\overline{aa-bb}|^2}$. To ſubtract $\frac{bb}{c}$ from $\frac{aa}{c}$, it will be wrote $\frac{aa-bb}{c}$. To ſubtract $a-\frac{xx}{m}$ from $\frac{yy}{m-n}$, it will be wrote $\frac{yy}{m-n}-a+\frac{xx}{m}$, which being reduced to a common denominator, if we think fit, will be $\frac{myy-amm+amn+mxx-nxx}{mm-mn}$. To ſubtract $\frac{b^4}{4a^2c-4a^2d}$ from $\frac{a^3+b^3}{2cd-2dd}$, it muſt be wrote $\frac{a^3+b^3}{2cd-2dd}-\frac{b^4}{4a^2c-4a^2d}$; and to reduce it to a common denominator, we muſt multiply a^3+b^3 by $2aa$, and $-b^4$ by d, and the whole muſt be divided by $4aacd-4aadd$; then it will be $\frac{2a^5+2aab^3-b^4d}{4aacd-4aadd}$.

Fractions how added and ſubtracted.

Multiplication of Fractions.

33. The numerators muſt be multiplied into one another, and alſo the denominators, and the new fraction will be the product of the fractions to be multiplied. Thus to multiply $\frac{ac}{b}$ into $\frac{bc}{d}$, the product will be $\frac{abcc}{bd}$, which is

Fractions how multiplied.

reduced

reduced to $\frac{acc}{d}$. To multiply $\frac{2ab}{b+c}$ into $\frac{3aa-bb}{5c}$, it will be wrote thus, $\frac{6a^3b-2ab^3}{5bc+5cc}$. The ſame muſt be done if there are integers with them, by conſidering an integer as a fraction, the denominator of which is unity. Thus to multiply $2a$, or $\frac{2a}{1}$, into $\frac{xx-3yy}{3x}$, the product will be $\frac{2axx-6ayy}{3x}$.

Let it be required to multiply $\frac{aa+bb}{a-b}$ into $a-b$. In this and the like caſes, becauſe the quantity which ought to multiply is the ſame as the denominator of the fraction, it will be ſufficient to expunge the denominator, and then the product will be $aa+bb$. If $aa-bb$ is to be multiplied into $\frac{aa-ab}{a+b}$, it may be obſerved, that $aa-bb$ is the ſame as $\overline{a+b} \times \overline{a-b}$, and therefore ſince it would be required to multiply $aa-ab$ into $a+b$ into $a-b$, and afterwards to divide by $a+b$; and becauſe $a+b$ would be a common diviſor both of the numerator and the denominator which would thence ariſe; the multiplication and diviſion by the ſame $a+b$ may be omitted, and it would be ſufficient to multiply the numerator by $a-b$, and the product will be $a^3-2aab+abb$. Thus the product of $\frac{a^3-abb}{xx-yy}$ into $\frac{a^3}{aa-bb}$ will be $\frac{a^4}{xx-yy}$.

Diviſion of Fractions.

Fractions how divided.

34. The Diviſion of Fractions is performed by multiplying croſs-wiſe, that is, by multiplying the numerator of the dividend by the denominator of the diviſor, which product muſt be the numerator of the fraction which is to be the quotient: and then multiplying the denominator of the dividend into the numerator of the diviſor, which product will be the denominator of the quotient. This quotient, if there is occaſion, muſt afterwards be reduced to the moſt ſimple expreſſion. Let it be required to divide $\frac{ab}{c}$ by $\frac{m}{n}$; the quotient will be $\frac{abn}{cm}$. Divide $\frac{ab}{c}$ by $\frac{-m}{n}$; the quotient will be $\frac{abn}{-cm}$, or $\frac{-abn}{cm}$; which is all one by § 13. Let it be required to divide $\frac{a^3-b^3}{a+b}$ by $\frac{aa-ab+bb}{c}$; it will be $\frac{a^3c-b^3c}{a^3+b^3}$.

It

It is eaſy to perceive, that if the two fractions, the dividend and diviſor, ſhall have the ſame denominator, it would be needleſs to multiply them croſs-wiſe. As if we were to divide $\frac{aa}{m}$ by $\frac{c-d}{m}$, in this caſe it would be enough to divide aa by $c-d$. For by multiplying croſs-wiſe it would be $\frac{aam}{cm-dm}$, and then reducing it to it's leaſt terms, it would be $\frac{aa}{c-d}$. Thus dividing $\frac{a^3-ab^2}{c-d}$ by $\frac{aa+2ab+bb}{c-d}$, the quotient would be $\frac{a^3-abb}{aa+2ab+bb}$; but by reduction, becauſe the numerator is $a \times \overline{a+b} \times \overline{a-b}$, and the denominator is $\overline{a+b} \times \overline{a+b}$, it will become $\frac{aa-ab}{a+b}$. After the ſame manner we muſt proceed when we are to divide an integer by a fraction, or a fraction by an integer; conſidering an integer as a fraction whoſe denominator is unity. Thus dividing the quantity $aa - xx$, or $\frac{aa-xx}{1}$, by $\frac{2yy-3xy}{3a}$, the quotient will be $\frac{3a^3-3axx}{2yy-3xy}$. And ſo of others.

Extraction of the Roots of Fractions.

35. The root of a fraction is extracted by extracting the root of the numerator, and then of the denominator, and the new fraction ariſing ſhall be the root of the fraction propoſed. So the ſquare-root of $\frac{aabb}{cc}$ will be $\frac{ab}{c}$. The ſquare-root of $\frac{a^4-2aabb+b^4}{aa+4ab+4bb}$ will be $\frac{aa-bb}{a+2b}$. The ſquare-root of $4aa + \frac{64xx-160ax}{25}$, that is of $\frac{100aa-160ax+64xx}{25}$, will be $\frac{10a-8x}{5}$. The ſame is to be underſtood of the cube-root, the biquadratick-root, and all others. Roots of fractions how extracted.

But now if the root cannot be extracted out of both the numerator and denominator, yet poſſibly it may be extracted out of one of the two. Let it be extracted out of which of the two it can, and before the other let the radical ſign be placed. Thus the cube-root of $\frac{a^6}{a^3-x^3}$ will be $\frac{aa}{\sqrt[3]{a^3-x^3}}$. The cube-root of $\frac{a^2x-x^3}{a^3b^3}$ will be $\frac{\sqrt[3]{aax-x^3}}{ab}$. And if the root cannot be extracted

neither

neither out of the numerator nor denominator, then the whole fraction muſt be included under the radical ſign. Thus the ſquare-root of $\frac{x^4 - a^4}{xx + bx}$ will be $\sqrt{\frac{x^4 - a^4}{xx + bx}}$.

Of the greateſt Common Diviſor of Two Quantities, or Formulas.

Greateſt common diviſor how found.

36. By a Formula I mean any analytical expreſſion whatever, whether complicate or not, the letters of which repreſenting indeterminate quantities, may be what we pleaſe; provided that whatever may be ſaid of that formula is to be underſtood as ſaid of any other, compounded of other letters, but ſimilar to the firſt.

To obtain the greateſt common diviſor of two quantities or formulas; in the firſt place it muſt be obſerved, that if every term of both is multiplied into the ſame quantity or number, in this caſe they muſt be divided by that quantity. Then each of the formulas muſt be ſet in order according to any letter at pleaſure; that is, that muſt be made the firſt term, in which that letter ariſes to the moſt dimenſions, and then the others in order. Let the two given formulas be $18a^3bx - 8a^4b - 3abx^3 - 8a^2bx^2 + bx^4$, and $6a^3b + bx^3 - abx^2 - 8a^2bx$; which becauſe they are diviſible by the letter b, let them be ſo divided, and then ſet in order (if you pleaſe) according to the letter x. They will be thus, $x^4 - 3ax^3 - 8a^2x^2 + 18a^3x - 8a^4$, and $x^3 - ax^2 - 8a^2x + 6a^3$. This being done, the firſt term, or that wherein the letter is of moſt dimenſions by which the terms are ſet in order, muſt be divided by the like term in the ſecond, namely x^4 divided by x^3 will give x in the quotient. Then the product of this quotient into the diviſor muſt be ſubtracted from the dividend, and we ſhall have the firſt remainder $-2ax^3 + 12a^3x - 8a^4$, which muſt be reduced to the moſt ſimple expreſſion, (as ought always to be done,) by dividing by $-2a$; then the remainder will be $x^3 - 6a^2x + 4a^3$. And becauſe the dimenſion of x in this remainder is the ſame as in the diviſor, by the ſaid diviſor this remainder muſt be divided; from whence in like manner muſt be ſubtracted the product of the quotient into the diviſor, and we ſhall have a ſecond remainder $ax^2 + 2a^2x - 2a^3$, or dividing by a it will be $x^2 + 2ax - 2a^2$. Now becauſe in this remainder the dimenſion of x is leſs than in the diviſor, the order muſt be inverted, and this remainder muſt be made the diviſor, and the firſt diviſor the dividend. And making the diviſion, the product of the quotient into the ſecond diviſor muſt be ſubtracted from the ſecond dividend, that is from $x^3 - ax^2 - 8a^2x + 6a^3$, and the remainder will be $-3ax^2 - 6a^2x + 6a^3$, which dividing by $-3a$ is $x^2 + 2ax - 2a^2$. Now whereas this laſt remainder is the ſame as the diviſor, it will be the greateſt common diviſor

divisor of the two formulas $x^4 - 3ax^3 - 8a^2x^2 + 18a^3x - 8a^4$, and $x^3 - ax^2 - 8a^2x + 6a^3$; which being multiplied into b, or $bx^2 + 2abx - 2a^2b$, will be the greatest common divisor of the two formulas at first proposed.

Let the two formulas be $x^4 - 4ax^3 + 11a^2x^2 - 20a^3x + 12a^4$, and $x^4 - 3ax^3 + 12a^2x^2 - 16a^3x + 24a^4$, being ordered according to the letter x. And as this is of the same dimensions in both, we are at liberty to take which of them we please for the divisor. Let the first therefore be divided by the second, and subtracting the product of the quotient into the divisor from the dividend, the first remainder will be $- ax^3 - a^2x^2 - 4a^3x - 12a^4$, which being divided by $- a$ is $x^3 + ax^2 + 4a^2x + 12a^3$. Here inverting the order, let this remainder be taken for the divisor, and the first divisor for the dividend. Then making the division, and subtracting the product of the quotient into this second divisor from the second dividend, the second remainder will be $- 4ax^3 + 8a^2x^2 - 28a^3x + 24a^4$, which being divided by $- 4a$ will be $x^3 - 2ax^2 + 7a^2x - 6a^3$. By the same second divisor let the division of this second remainder be continued, and making the subtraction as usual, we shall have a third remainder $- 3axx + 3a^2x - 18a^3$, or dividing by $- 3a$ it will be $x^2 - ax + 6a^2$. Let the order be again inverted, and let the second divisor be divided by this third remainder $x^3 + ax^2 + 4a^2x + 12a^3$, and making the subtraction as usual, the remainder will be found to be $2ax^2 - 2a^2x + 12a^3$; or dividing by $2a$, it will be $xx - ax + 6aa$, the same quantity as that which was a divisor before, and which is therefore the greatest common divisor of the two proposed quantities.

Let the two formulas be $f^4 - aaff - bbff + aabb$, and $f^3 - aff - 2abf + 2a^2b$, which are ordered according to the letter f. Let the first be divided by the second, and the product of the quotient into the divisor being subtracted from the dividend, will give the first remainder $af^3 - a^2f^2 + 2abff - bbff - 2a^2bf + a^2b^2$. And if we go on to divide by the same divisor, and the product of the divisor into the quotient being subtracted from the dividend, we shall have a second remainder $2abff - b^2ff - 2a^3b + a^2b^2$, or dividing by b it will be $2aff - bff - 2a^3 + a^2b$. Then invert the order, and divide the first divisor by this second remainder, and taking the product of the quotient $\frac{f}{2a-b}$ into the said remainder, which has now served as a divisor, and then making the subtraction, we shall have a third remainder $- aff + a^2f - 2abf + 2a^2b$, or dividing by $- a$, it is $ff - af + 2bf - 2ab$. The division is to be continued in the same order, and the product of the quotient $\frac{1}{2a-b}$ into the divisor $2aff - bff + a^2b - 2a^3$ being subtracted, we shall have a fourth remainder $- af + 2bf - 2ab + a^2$, by which, inverting the order, the third remainder must be divided, and the product of the quotient $\frac{f}{2b-a}$ into the di-

 visor

visor being subtracted, we shall have a fifth remainder $2bf - 2ab$, or dividing by $2b$, it is $f - a$. Now if this is divided by the fourth remainder $-af + 2bf - 2ab + a^2$, and the product of the quotient $\frac{1}{2b-a}$ into the divisor is subtracted, nothing will remain. Whence if by the denominator of the last quotient, it being a fraction, the last divisor $-af + 2bf - 2ab + a^2$ shall be divided, the quotient will be $f - a$, the greatest divisor of the two quantities proposed. But because it was at pleasure whether we chose for a divisor that which was made the dividend, or *vice versâ*; that is, we might have divided $-af + 2bf - 2ab + aa$ by $f - a$; let the division be actually made, and the quotient will be $2b - a$ without a remainder; and therefore $f - a$ will be the greatest common divisor, as found above by means of the other division.

Wherefore two formulas may have a greatest common divisor, though being ordered according to some certain letter, it cannot be found in this manner; in which case it must be set in order again, according to some other of it's letters. Now if this be tried by setting it in order according to any other letter, and if it will not then succeed, the quantities proposed will have no greatest common divisor. Thus it would not be found in the last example, by setting them in order according to the letter b; which however is found by ordering them according to the letter f.

Now the fraction $\frac{x^4 - 3ax^3 - 8a^2x^2 + 18a^3x - 8a^4}{x^3 - ax^2 - 8a^2x + 6a^3}$ being given, if we divide the numerator and denominator by $x^2 + 2ax - 2a^2$, we shall have the fraction $\frac{x^2 - 5ax + 4a^2}{x - 3a}$.

Also the fraction $\frac{x^4 - 4ax^3 + 11a^2x^2 - 20a^3x + 12a^4}{x^4 - 3ax^3 + 12a^2x^2 - 16a^3x + 24a^4}$, by dividing by $x^2 - ax + 6a^2$, will become $\frac{x^2 - 3ax + 2a^2}{x^2 - 2ax + 4a^2}$.

And the fraction $\frac{f^4 - a^2f^2 - b^2f^2 + a^2b^2}{f^3 - af^2 - 2abf + 2a^2b}$, by dividing by $f - a$, will become $\frac{f^3 + af^2 - b^2f - ab^2}{f^2 - 2ab}$.

Thus these fractions are reduced to more simple expressions, as is said above at § 30.

Reduction

Reduction of Irrational Quantities to more simple Expressions.

37. It has been observed already, how irrational quantities arise, which are otherwise called Surds, or Radicals. For when the root required cannot be actually extracted, then we have recourse to a *radical vinculum*, which insinuates it. But it often happens that the quantity under the vinculum is the product of two factors, one of which is a true power of the same name as the root required. As if it were $\sqrt{aabc}$, or $\sqrt{a^2b - a^2x}$; the first of which is the product of aa into bc, and the other is the product of aa into $b - x$. Thus also $\sqrt[3]{a^3x - a^3y}$ is the cube-root of the product of a^3 into $x - y$. In this case the root may be extracted out of such of the factors as will admit it, and wrote without the radical sign, and the other factor may remain under the sign. And this is called extracting the root in part, or reducing the radical to a more simple expression. Thus $\sqrt{aabc}$ will be reduced to $a\sqrt{bc}$. And $\sqrt{aab - aax}$ will be the same as $a\sqrt{b - x}$; $\sqrt[3]{a^3x - a^3y}$ will be reduced to $a\sqrt[3]{x - y}$; and so of others. In like manner, because $\sqrt{48aabc}$ is the root of the product of $16aa$ into $3bc$, it will be reduced to $4a\sqrt{3bc}$. Thus, because $\sqrt{\frac{a^3b - 4a^2b^2 + 4ab^3}{cc}}$ is the root of the product of $\frac{aa - 4ab + 4bb}{cc}$ into ab, and the root of $\frac{aa - 4ab + 4bb}{cc}$ is $\frac{a - 2b}{c}$; the root reduced will be $\frac{a - 2b}{c}\sqrt{ab}$. Thus the root $\sqrt{\frac{a^2m^2x^2 + 4a^2m^3p}{p^2z^2}}$, when reduced, will be $\frac{am}{pz}\sqrt{x^2 + 4mp}$. And the root $\sqrt[3]{8a^3b + 16a^4}$ will be $2a\sqrt[3]{b + 2a}$. Thus $\sqrt{a^3 - 3a^2b + 3ab^2 - b^3}$, which is the root of the product of $aa - 2ab + bb$ into $a - b$, will be reduced to $\overline{a - b} \times \sqrt{a - b}$. But very often it cannot be known by inspection only, what are the factors from whence the proposed radical proceeds. In which case we must have recourse to the method of finding all the divisors, which I shall give in it's proper place; and if among these shall be one, which is exactly a power with the same exponent as the radical indicates; the proposed quantity may then be reduced in the manner now explained. Surds reduced how.

Reduction of Radicals to the same Denomination.

38. Those are called radicals of a different denomination which have a different index or exponent. To reduce them therefore to radicals of the same index, we must proceed thus. If the index of one of the radicals is an aliquot part of the index of the other, the greater index must be divided by the lesser, Radicals how reduced to the same denomination.

 and

and the quotient ſhows that power, to which the quantities muſt be raiſed which are under the radical of the leſſer index, and to which muſt be prefixed the radical of the greater index. Let it be propoſed to reduce to the ſame index the quantities $\sqrt{\sqrt{ax}}$ and $\sqrt{a}$; or which is the ſame, $\sqrt[4]{ax}$ and $\sqrt[2]{a}$. Becauſe 4 divided by 2 gives 2 for the quotient, therefore the quantity a of the leſſer index muſt be raiſed to it's ſquare, which is aa, and it will be $\sqrt[4]{aa}$, and therefore is reduced to the ſame index or denomination as $\sqrt[4]{ax}$. Thus $\sqrt[6]{a^3b^3 + ab^5}$ and $\sqrt{ab}$ will make $\sqrt[6]{a^3b^3 + ab^5}$ and $\sqrt[6]{a^3b^3}$. But if one of the exponents is not an aliquot part of the other, the leaſt number muſt be found which is diviſible without a fraction by each of the exponents of the given radicals, and this will be the index of the common radical. Then the quantities muſt be raiſed to the next inferior degree of the number, by which the exponents are increaſed of the reſpective radicals, and then to the powers ſo raiſed let the common radical now found be prefixed. Let the two quantities $\sqrt[2]{aq}$ and $\sqrt[3]{aaq}$ be given, to be reduced to a common radical. The leaſt number diviſible by 2 and by 3 will be 6, and therefore $\sqrt[6]{}$ will be the common radical. Now, becauſe the index of the ſquare-root is in this caſe increaſed by 4, and that of the cube-root by 3; therefore the firſt will become $\sqrt[6]{a^3q^3}$, and the ſecond will be $\sqrt[6]{a^4qq}$. If the radicals to be reduced are more than two, any two are to be reduced firſt, then the third, and ſo on ſucceſſively.

The manner of reducing rationals to any radical, is plain of itſelf, without the aſſiſtance of rules; by raiſing the rational to any power of the ſame name or index of the radical given, and then prefixing to it the ſame radical.

Addition and Subtraction of Radical Quantities.

Surds how added or ſubtracted.

39. To add them together, the radical quantities are wrote one after another with their proper ſigns. And to ſubtract them, the ſigns of thoſe to be ſubtracted are to be changed, as is done in other quantities. Thus to add $5a\sqrt{bc}$ to $2b\sqrt{bx}$ to $-c\sqrt{zy}$, they muſt be wrote thus, $5a\sqrt{bc} + 2b\sqrt{bx} - c\sqrt{zy}$. To add $5x\sqrt{ab}$ to $3x\sqrt{ab}$ to $y\sqrt{bx}$, they muſt be wrote thus, $5x\sqrt{ab} + 3x\sqrt{ab} + y\sqrt{bx}$; and then reducing like terms, which ought always to be done, they will become $8x\sqrt{ab} + y\sqrt{bx}$. To add $a - b$ to $\sqrt{aa - xx}$, it muſt be wrote $a - b + \sqrt{aa - xx}$. And the ſame is to be done in ſubtraction, having regard to the ſigns.

Multi-

Multiplication of Irrational Quantities.

40. To multiply rational quantities by ſurds or radicals, the rational is wrote together with the radical, without any ſign between, only prefixing to the product ſuch ſign, whether poſitive or negative, as ſhall be required by the common rules of multiplication; and this is to be underſtood always to be done. Therefore the product of a into $\sqrt{aa - xx}$ will be $a\sqrt{aa - xx}$. The product of ab into $-\sqrt{ab}$ will be $-ab\sqrt{ab}$. And if the rational quantities or radicals ſhall conſiſt of ſeveral terms, or if they are complicate, every term of one muſt be multiplied into every term of the other. Wherefore the product of $aa - xx$ into $\sqrt{xx - yy}$ will be $\overline{aa - xx}\sqrt{xx - yy}$, where it is underſtood, that all thoſe terms are multiplied into the radical, which are under the *vinculum*. Surds how multiplied.

41. To multiply radicals among themſelves, ſuppoſing them to be of the ſame denomination, or reduced to ſuch, the quantities muſt be multiplied into each other which are under the radical ſigns, and to the product muſt be put the ſame radical vinculum, with ſuch a ſign, either poſitive or negative, as the common rule requires. Thus to multiply $\sqrt{bc}$ into $\sqrt{xy}$, the product will be $\sqrt{bcxy}$. To multiply $\sqrt{\frac{aa - xx}{x}}$ into $-\sqrt{aa + xx}$, the product will be $-\sqrt{\frac{a^4 - x^4}{x}}$. Surds multiplied by ſurds.

42. Moreover, if the radicals ſhall have rational co-efficients, whether numeral or literal, thoſe co-efficients muſt be multiplied together, and alſo the radicals together, and the product of the co-efficients muſt be put before the radical, without any ſign between. Thus $a\sqrt[3]{bbc}$ into $a\sqrt[3]{bxx}$ will be $aa\sqrt[3]{b^3cx^2}$, which reduced is $aab\sqrt[3]{cxx}$. So $2a - \sqrt{aa - xx}$ into $\frac{b}{a}\sqrt{aa + xx}$ will be $2b\sqrt{aa + xx} - \frac{b}{a}\sqrt{a^4 - x^4}$. When they have rational co-efficients.

43. According to this rule, to multiply $m\sqrt{ab}$ into $n\sqrt{ab}$, the product would be $mn\sqrt{aabb}$. But $aabb$ is a ſquare whoſe root is ab, and therefore the product will be $mnab$. So that, to multiply two like quadratick radicals into each other, it will ſuffice to take away the radical vinculum, and the quantities which were under it, multiplied into the product of the co-efficients, will be the total product. Thus $\frac{2b}{a}\sqrt{ax - xx}$ into $-\frac{c}{3}\sqrt{ax - xx}$ will be $-\frac{2bc}{3a} \times \overline{ax - xx}$, that is, $-\frac{2}{3}bcx + \frac{2bcxx}{3a}$. But here it muſt be obſerved, that if the radicals having Sometimes may become rationals.

having no co-efficients, or unity only, are affected by the fame fign, pofitive or negative, the *vinculum* being taken away, the quantities muft be left with the fign they have. And if the radicals have contrary figns, all the figns of the quantity muft be changed. For example, $\sqrt{\frac{aa - xx}{x}}$ into $\sqrt{\frac{aa - xx}{x}}$, or elfe $-\sqrt{\frac{aa - xx}{x}}$ into $-\sqrt{\frac{aa - xx}{x}}$, will be $\frac{aa - xx}{x}$. Alfo $\sqrt{\frac{aa - xx}{x}}$ into $-\sqrt{\frac{aa - xx}{x}}$ will be $\frac{-aa + xx}{x}$, or $\frac{aa - xx}{-x}$. The reafon of which is, becaufe $\sqrt{\frac{aa - xx}{x}}$, (and fo of any other,) is always underftood to have $+ 1$ for it's co-efficient, and $-\sqrt{\frac{aa - xx}{x}}$ to have $- 1$. Therefore the product ought to be $1 \times \frac{aa - xx}{x}$ in the firft cafe, and $- 1 \times \frac{aa - xx}{x}$ in the fecond. Here are other examples of thefe multiplications.

$\sqrt{ab} + \sqrt{aa - xx}$ into $\sqrt{ab} + \sqrt{aa - xx}$ makes the product $ab + \sqrt{a^3b - abx^2} + aa - xx + \sqrt{a^3b - abx^2}$, or $ab + a^2 - x^2 + 2\sqrt{a^3b - abx^2}$.

$x - \sqrt{\frac{\sqrt{4a^4 + y^4} - y^2}{2}}$ into $x + \sqrt{\frac{\sqrt{4a^4 + y^4} - y^2}{2}}$ makes the product $xx - x\sqrt{\frac{\sqrt{4a^4 + y^4} - y^2}{2}} - \frac{\sqrt{4a^4 + y^4} - y^2}{2} + x\sqrt{\frac{\sqrt{4a^4 + y^4} - y^2}{2}}$, that is, $xx + \frac{1}{2}yy - \frac{\sqrt{4a^4 + y^4}}{2}$.

$\sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}pp}}$ into $\sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}pp}}$ makes the product $\sqrt[3]{\frac{1}{4}qq + q\sqrt{\frac{1}{4}qq - \frac{1}{27}pp} + \frac{1}{4}qq - \frac{1}{27}pp}$, that is, $\sqrt[3]{\frac{1}{2}qq - \frac{1}{27}pp + q\sqrt{\frac{1}{4}qq - \frac{1}{27}pp}}$.

Rational co-efficients how brought under the vinculum.

44. Becaufe $a\sqrt{ax}$, $\overline{a - b} \times \sqrt{ax - xx}$, and fuch others, are the products of a rational quantity into a radical, and we already know how to reduce any rational to any radical we pleafe; we can always make the rational multiplier to pafs under the *vinculum* without any alteration of the quantity. Thus $a\sqrt{a - x}$ will be the fame as $\sqrt{a^3 - a^2x}$; $\overline{a - b} \times \sqrt{xy}$ will become $\sqrt{a^2xy - 2abxy + b^2xy}$; $ax\sqrt[3]{m - n}$ will be $\sqrt[3]{ma^3x^3 - na^3x^3}$; and fo of any others.

Different furds how multiplied.

45. If the radicals to be multiplied are not of the fame name, they may be reduced to fuch, and then the multiplication may be made as before. But very often it will be more commodious to infinuate it only, without actually performing it, and this by writing one radical after another, without any fign interpofed, except the mark of multiplication. Thus $\sqrt{aa - xx} \times \sqrt[3]{xxy}$ will denote the product of thefe two radicals.

Divifion

Division of Radical Quantities.

46. In every term of the dividend and of the divisor, if the same radical is found, omitting this, the rational quantities are to be divided as usual, and what results will be the quotient. Thus to divide $5a\sqrt{3}$ by $3a\sqrt{3}$, the quotient will be $\frac{5}{3}$. To divide $6\sqrt{a^4 + a^2b^2}$ by $2\sqrt{a^2b^2 + b^4}$, or $6a\sqrt{a^2 + b^2}$ by $2b\sqrt{a^2 + b^2}$, the quotient will be $\frac{3a}{b}$. To divide $aa\sqrt[4]{aa + xx} - 2ax\sqrt[4]{aa + xx} + xx\sqrt[4]{a^2 + x^2}$ by $a\sqrt[4]{aa + xx} - x\sqrt[4]{aa + xx}$, omitting the radical, and dividing $aa - 2ax + xx$ by $a - x$, the quotient will be $a - x$. To divide $aa + bb$ by $\sqrt{aa + bb}$, because the dividend is $\sqrt{aa + bb} \times \sqrt{aa + bb}$, the quotient will be $\sqrt{aa + bb}$. How surds are to be divided.

47. But when the radicals are not the same, though they have the same exponent of the root; let the quantities under the *vinculum* be divided by the rational quantities in the usual manner, and to the quotient prefix the common *vinculum*. Thus to divide $\sqrt[3]{a^3b - ab^3}$ by $\sqrt[3]{aa - bb}$, dividing $a^3b - ab^3$ by $a^2 - b^2$ there arises ab, and therefore the quotient required is $\sqrt[3]{ab}$. When the index is the same, but the surds are different.

48. And if the exponents of the roots are different, they may be reduced to the same, and then the operation will be as before. Thus to divide $\sqrt{a^4 + 2a^3b - 2ab^3 - b^4}$ by $a + b$, the square of $a + b$ must be found, and put under the *vinculum*, which will be then $\sqrt{aa + 2ab + bb}$. Then by the quantity under this *vinculum* the other quantity must be divided, and the result will be $aa - bb$. Therefore the quotient required will be $\sqrt{aa - bb}$. When the index also is different.

By combining these rules with those of common division, quantities still more complicate may be divided. Thus to divide $a^3b - ab^2c - a^2b\sqrt{bc} + b^2c\sqrt{bc}$ by $a - \sqrt{bc}$, it may be performed as is usual in division.

Dividend.	$a^3b - ab^2c - a^2b\sqrt{bc} + b^2c\sqrt{bc}$.	Divisor	$a - \sqrt{bc}$
Rem.	$- ab^2c \qquad - b^2c\sqrt{bc}$	Quotient	$a^2b - b^2c$

Thus dividing $a^3 - abc + a^2\sqrt{bc} - bc\sqrt{bc}$ by $a - \sqrt{bc}$, the quotient will be $aa + bc + 2a\sqrt{bc}$. And when the division will not succeed, the quantities must be wrote in form of a fraction.

Extraction

Extraction of the Square-Root of Radical Quantities.

The square-root of surds extracted.

49. When quantities any how compounded of rationals and radicals are quadratick radicals, the rule for extracting the square-root will be this. Taking such a part of the quantity proposed as is greater than the remaining part, from the square of this greater part let the square of the lesser part be subtracted, and to the greater part let the square-root of the remainder be added, and likewise be subtracted from it. The square-root of the half of this sum, and of the half of this difference, being taken together, and taking the same sign to this second as belongs to the minor part, will make the square-root of the proposed quantity. Thus let us extract the square-root of the quantity $3 + \sqrt{8}$; subtracting the square of $\sqrt{8}$ from the square of 3, there will remain 1, the root of which is also 1. Adding this therefore to the greater part, or 3, they will make 4, and subtracting it from the same, it will make 2; now the square-root of the half of 4 is $\sqrt{2}$, and the square-root of the half of 2 is 1; therefore $\sqrt{2} + 1$ will be the root required.

If we would have the square-root of $6 + \sqrt{8} - \sqrt{12} - \sqrt{24}$; from the square of $6 + \sqrt{8}$ subtracting the square of $-\sqrt{12} - \sqrt{24}$, there remains 8, the root of which $\sqrt{8}$ being added to $6 + \sqrt{8}$, the greater part, will make $6 + 2\sqrt{8}$, and subtracted from the same greater part will make 6. Therefore the first part of the root required will be $\sqrt{\frac{6 + 2\sqrt{8}}{2}}$, that is, $\sqrt{3 + \sqrt{8}}$, and the second part will be $-\sqrt{\frac{6}{2}}$, that is $-\sqrt{3}$, (for the lesser part of the proposed quantity was affected by the negative sign;) whence $\sqrt{3 + \sqrt{8}} - \sqrt{3}$ will be the root required. But by the last example it may be seen, that $\sqrt{3 + \sqrt{8}}$ is the same as $1 + \sqrt{2}$; therefore, lastly, the root of the quantity proposed will be $1 + \sqrt{2} - \sqrt{3}$.

Let us extract the square-root of $aa + 2x\sqrt{aa - xx}$. Taking from the square of aa the square of $2x\sqrt{aa - xx}$, there will remain $a^4 - 4aaxx + 4x^4$, the root of which is $aa - 2xx$. This added to the greater part aa, and taking the half of it, will make $aa - xx$: and subtracted from the same, and taking half the difference, will make xx. Therefore the root required is $\sqrt{aa - xx} + x$.

Let us extract the square root of the quantity $aa + 5ax - 2a\sqrt{ax + 4xx}$. From the square of $aa + 5ax$, the greater part, subtracting the square of $-2a\sqrt{ax + 4xx}$, there will remain $a^4 + 6a^3x + 9a^2x^2$, the root of which is $aa + 3ax$. This added to the greater part, and taking it's half, it will be $aa + 4ax$; and subtracting and taking the half, it will be ax. Therefore the root required will be $\sqrt{aa + 4ax} - \sqrt{ax}$.

To

To extract the ſquare-root of this quantity $a\sqrt{bc} + d\sqrt{bc} + 2\sqrt{abcd}$. From the ſquare of $a\sqrt{bc} + d\sqrt{bc}$ ſubtracting the ſquare of $2\sqrt{abcd}$, there remains $aabc - 2abcd + bcdd$, the root of which is $a\sqrt{bc} - d\sqrt{bc}$; which being added to the major part, and ſubtracted from the ſame, and taking half of the ſum and difference, the half of the ſum will be $a\sqrt{bc}$, and half of the difference $d\sqrt{bc}$. Therefore the root required is $\sqrt{a\sqrt{bc}} + \sqrt{d\sqrt{bc}}$, that is, $\sqrt{\sqrt{aabc}} + \sqrt{\sqrt{bcdd}}$, or $\sqrt[4]{aabc} + \sqrt[4]{bcdd}$. If the root cannot be extracted, the quantity muſt be put under a radical *vinculum*, as uſual.

The Calculation of Powers.

50. There is nothing now to be obſerved concerning the Addition or Subtraction of Powers; they are to be written one after another with their proper ſigns in the firſt caſe, and in the ſecond by changing the ſigns of the quantities to be ſubtracted. But as to the other operations which belong to their exponents, it may be firſt obſerved, that, taking unity for the firſt term, and any quantity whatever, as a, for the ſecond, and then ſucceſſively the other powers of the ſame quantity a in order, it is plain we ſhall form an increaſing geometrical progreſſion, $1, a, a^2, a^3, a^4$, &c.; and that the exponents of this progreſſion will form an arithmetical progreſſion increaſing, which will be 0, 1, 2, 3, 4, 5, &c. The firſt term of this is 0, becauſe unity being the firſt term in the geometrical progreſſion, in this the quantity a is raiſed to no power; for $1 = \frac{a}{a} = a^0$. Wherefore, multiplying either $\frac{a}{a}$, or a^0, by a, which does not deſtroy the equality, the product will be $a = a^{0+1}$, which are magnitudes plainly identical. And beſides, if we continue the ſame geometrical progreſſion below unity, it will be $1, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4}$, &c. And likewiſe, continuing the arithmetical progreſſion of the exponents, they will become $0, -1, -2, -3, -4$, &c. And therefore the exponents of ſuch powers will be negative. So that $\frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}$, &c. will be the ſame as a^{-1}, a^{-2}, a^{-3}, &c. And in general, $\frac{1}{a^n}$ will be the ſame as a^{-n}; that is to ſay, we may always make a power to paſs into the numerator of a fraction out of the denominator, and *vice verſâ*, only by changing the ſign of the index. Powers how calculated when the exponents are integers.

51. Moreover, if we ſhould deſire to introduce new intermediate terms into the geometrical progreſſion, the exponents of theſe would alſo be intermediate terms When they are fractions.

terms in the arithmetical progreffion, analogous to the former. So, becaufe $\sqrt{a}$ is a geometrical mean between unity and a, the exponent of this ought to be an arithmetical mean between o and unity, and therefore muft be $\frac{1}{2}$; fo that $a^{\frac{1}{2}}$ will be the fame as $\sqrt{a}$. If two mean proportionals are interpofed between 1 and a, of which the firft will be $\sqrt[3]{a}$, and the fecond $\sqrt[3]{aa}$, there muft be two arithmetical means between o and 1, which are $\frac{1}{3}$ and $\frac{2}{3}$; fo that $a^{\frac{1}{3}}$ will be the fame as $\sqrt[3]{a}$, and $a^{\frac{2}{3}}$ will be the fame as $\sqrt[3]{aa}$. If three mean proportionals are introduced, they will be $\sqrt[4]{a}$ the firft, $\sqrt[4]{aa}$ the fecond, and $\sqrt[4]{aaa}$ the third, and their exponents will be $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$; therefore $\sqrt[4]{a}$ will be the fame as $a^{\frac{1}{4}}$, and $\sqrt[4]{aa}$ the fame as $a^{\frac{2}{4}}$, or $a^{\frac{1}{2}}$, and $\sqrt[4]{a^3}$ will be the fame as $a^{\frac{3}{4}}$. And thus we may proceed to as many mean proportionals as we pleafe; fo that, in general, it will be $\sqrt[m]{a^n}$, the fame as $a^{\frac{n}{m}}$.

The fame things obtain in refpect of the progreffion produced by defcending below unity. Thus, as $\frac{1}{\sqrt{a}}$ is a mean proportional between unity and $\frac{1}{a}$, or between unity and a^{-1}, fo it's index fhould be an arithmetical mean between o and -1, that is $-\frac{1}{2}$; therefore $\frac{1}{\sqrt{a}}$ will be the fame as $a^{-\frac{1}{2}}$, or $\frac{1}{a^{\frac{1}{2}}}$. Thus likewife $\frac{1}{\sqrt[3]{a}}$ and $\frac{1}{a^{\frac{1}{3}}}$ and $a^{-\frac{1}{3}}$ will be the fame. And $\frac{1}{\sqrt[3]{aa}}$, $\frac{1}{a^{\frac{2}{3}}}$, $a^{-\frac{2}{3}}$ will be the fame. And fo, in general, $\frac{1}{\sqrt[m]{a^n}}$, $\frac{1}{a^{\frac{n}{m}}}$, and $a^{-\frac{n}{m}}$ will be the fame.

And what has been faid concerning integral or fractional powers of fimple quantities, is to be underftood alfo of compound quantities. Thus, for example, $\frac{1}{\overline{aa+bb}|^{n}}$ is the fame as $\overline{aa+bb}|^{-n}$. So $\sqrt[m]{\overline{aa+bb}|^{n}}$ will be the fame as $\overline{aa+bb}|^{\frac{n}{m}}$; and the like of others.

Powers how multiplied or divided.

52. From the nature of the two foregoing progreffions, the geometrical and arithmetical, we obtain a method for the multiplication or divifion of any two powers of the fame quantity, whatever they may be; and that is, by adding the exponents together when the powers are to be multiplied, and by fubtracting

ing the exponent of the divifor from that of the dividend, when the powers are to be divided. For, as to multiplication, as the product is the fourth proportional from unity and the two factors, thefe four terms will be in a geometrical proportion, and their exponents in an arithmetical progreffion. Therefore the exponent of the fourth, that is of the product, muft be greater than the exponent of the third, by as much as the exponent of the fecond is greater than the exponent of the firft. But the exponent of the fecond is greater than the exponent of the firft, which is o, by it's whole quantity; therefore the exponent of the fourth ought to be greater than the exponent of the third by the whole exponent of the fecond; that is, it ought to be equal to the fum of the exponents of the fecond and third. As to divifion, it has the fame proportion as multiplication, but only inverted. It's firft term is the dividend, it's fecond the divifor, the third the quotient, and the fourth is unity. Therefore as much as the exponent of the dividend is greater than the exponent of the divifor, fo much the exponent of the quotient ought to be greater than o. Therefore it ought to be exactly the difference of the exponents of the dividend and the divifor.

So that to multiply aa by a, or a^2 by a^1, the product will be a^{2+1} or a^3. To multiply a^3 into a^2, the product is a^{3+2}, or a^5. To multiply a^6 into a^{-2}, the product is a^{6-2}, or a^4. To multiply $a^{\frac{1}{2}}$ into $a^{\frac{1}{3}}$, the product is $a^{\frac{1}{2}+\frac{1}{3}}$, that is $a^{\frac{5}{6}}$. To multiply $a^{-\frac{2}{3}}$ into $a^{\frac{1}{5}}$, the product is $a^{-\frac{2}{3}+\frac{1}{5}}$, that is $a^{-\frac{7}{15}}$. To multiply $a^{\pm\frac{n}{m}}$ into $a^{\pm\frac{r}{t}}$, the product is $a^{\pm\frac{n}{m}\pm\frac{r}{t}}$, or $a^{\frac{\pm nt \pm mr}{mt}}$.

And fo to divide a^3 by a^1, the quotient will be a^{3-1}, or a^2. To divide a^5 by a^{-2}, the quotient will be a^{5+2}, or a^7. To divide a^2 by $a^{\frac{1}{2}}$, the quotient will be $a^{2-\frac{1}{2}}$, or $a^{\frac{3}{2}}$. To divide $a^{\frac{2}{3}}$ by $a^{-\frac{1}{2}}$, the quotient will be $a^{\frac{2}{3}+\frac{1}{2}}$, or $a^{\frac{7}{6}}$. To divide $a^{\pm\frac{n}{m}}$ by $a^{\pm\frac{r}{t}}$, the quotient will be $a^{\pm\frac{n}{m}\mp\frac{r}{t}}$, that is, $a^{\frac{\pm nt \mp mr}{mt}}$.

53. And becaufe in the progreffion before confidered, taking any term whatever, the fame term with a double exponent will be the fquare of the term fo taken; and a term with a treble exponent will be the cube of the affumed term; and a term with a quadruple exponent will be the fourth power; and fo on. And a term with half the exponent will be the fquare-root of the term affumed; a term with a third part, a fourth part, &c. will be the cube-root, the fourth root, &c. of the term affumed. It follows therefore that, to reduce one power

Powers may be raifed, or roots extracted, by the exponents.

power to another, it will be ſufficient to multiply the exponent of the given power by the exponent of that power to which we would raiſe it : and to extract any root, it will be enough to divide it's index by the index of the given root. Thus to raiſe a^2 to it's cube, it will be $a^{2 \times 3}$, or a^6. To raiſe $a^{\frac{2}{3}}$ to the cube, it will be $a^{\frac{2}{3} \times 3}$, or a^2. To raiſe $a^{-\frac{1}{4}}$ to the fifth power, it will be $a^{-\frac{1}{4} \times 5}$, or $a^{-\frac{5}{4}}$. To raiſe $a^{\pm \frac{n}{m}}$ to the power whoſe index is $\pm \frac{r}{t}$, it will be $a^{\pm \frac{nr}{mt}}$. Thus to extract the ſquare-root of a^5, it will be $a^{\frac{5}{2}}$. To extract the cube-root of $a^{\frac{1}{2}}$, it will be $a^{\frac{1}{6}}$. To extract the root r of $a^{\pm \frac{m}{n}}$, it will be $a^{\pm \frac{m}{nr}}$, &c.

Extended to compound quantities.

54. What I have here ſaid concerning the powers or roots of one and the ſame ſimple quantity, may be underſtood in like manner concerning the powers or roots of any compound quantities, as is evident. And by this method the calculus of fractions and radicals will be much facilitated.

Of Linear or Simple Diviſors of any Formula whatever.

Simple diviſors how found ; as alſo compound diviſors.

55. Any quantity or formula whatever, whether complicate or not, is ſaid to be *prime* or *ſimple*, when it is not exactly diviſible by any other quantity, except itſelf or unity. And it is called *compound* when it is exactly diviſible by ſome other quantity. Thus, for example, $a + b$, $aa + xx$, $x^3 - aax + aab$, and ſuch others, will be prime or ſimple. But ab is compound, becauſe diviſible by a or b. So $aa - xx$ is compound, becauſe diviſible by $a + x$ or $a - x$. And ſo of others.

Two or more formulas are *relative primes*, when they have no common diviſor, and that the leſſer is not a diviſor of the greater. Such between themſelves will be aa and bb. Alſo $aa + 2ab + bb$ and $aa + bb$, &c. And on the contrary, they are abſolutely and relatively compound, between themſelves, when they have ſome common diviſor, or that one of them can divide the other. Such are aa and ab, which are both diviſible by a ; ſuch are $aa - xx$ and $a + x$, which are diviſible by $a + x$, &c.

In order to have all the ſimple diviſors of any quantity, either numeral, or literal, or mixt, it muſt be divided by the leaſt of it's diviſors, and the quotient again by the leaſt of it's diviſors, and ſo on continually till a quotient ariſes, which

which can no longer be divided except by itself. The quantities thus arising, unity being comprehended among them, will be all the simple divisors. And if they are taken two by two, three by three, and so on; according to all the combinations possible, they will give likewise all the compound divisors.

For example, let us find all the divisors, simple or compound, of the number 300. Let the given number 300 be wrote at A, and at one side, as at B, set down it's least divisor, as 2. Then dividing by 2, write the quotient 150 at A under 300; and again divide this number 150 by 2, and over against it at B write the divisor 2, and the quotient 75 at A under the first quotient 150. Now, because 75 is not divisible by 2, let it be divided by 3, and write the divisor 3 over against it at B, and under it at A the quotient 25. The least divisor of 25 will be 5, which must be wrote over against it at B, and the quotient 5 under it at A. The last quotient 5 is not divisible unless by itself; therefore it must be wrote aside at B, and we shall have all the prime divisors; to which we may add unity, because it is always a divisor of any quantity. Now to have all the compound divisors, according to all the combinations, let the first and second divisors be multiplied together, and the product 4 be wrote at B over against the second divisor. By the third divisor let all above it be multiplied, and let the products 6, 12, be wrote aside, setting down but once those that may chance to be repeated. In like manner, by the fourth let all above it be multiplied, and the products set down as before: and so on successively to the last. Now the numbers wrote at B will be all the divisors of the proposed number 300.

A.	B.						
	1						
300	2						
150	2	4					
75	3	6	12				
25	5	10	15	20	30	60	
5	5	25	50	75	100	150	300
1							

Let the given formula be $21abb$, of which we are to find all the divisors. As it is not divisible by 2, let it be divided by 3, which is to be wrote over against it at B, and the quotient $7abb$ under it at A. Let $7abb$ be divided by 7, which is to be wrote over against it, and the quotient abb underneath. Let abb be divided by a, which is wrote aside, and the quotient bb under it. Then divide bb by b, which is wrote aside, and the quotient b underneath. This is to be divided by b, and wrote over against it; and then we shall have all the prime divisors 1, 3, 7, a, b, b, of the proposed quantity. To have those that are compound we must multiply 3 into 7, and the product is 21. Multiply 3, 7, 21 into a, and the products are $3a$, $7a$, $21a$. Multiply all the divisors 3, 7, 21, a, $3a$, $7a$, $21a$ into b, and there will arise $3b$, $7b$, $21b$, ab, $3ab$, $7ab$, $21ab$; and

and fo proceed. Thus the column B will contain all the divifors of the quantity propofed, both fimple and compound.

A.	B.								
	1								
$21abb$	3								
$7abb$	7	21							
abb	a	$3a$	$7a$	$21a$					
bb	b	$3b$	$7b$	$21b$	ab	$3ab$	$7ab$	$21ab$	
b	b	bb	$3bb$	$7bb$	$21bb$	abb	$3abb$	$7abb$	$21abb$
1									

In like manner, let $2abb - 6aac$ be given. Let it firft be divided by 2, and the quotient $abb - 3aac$ by a, and the new quotient $bb - 3ac$ by itfelf, as being divifible by no other quantity. And therefore all the divifors will be as in the column B.

A.	B.
	1
$2abb - 6aac$	2
$abb - 3aac$	a, $2a$
$bb - 3ac$	$bb - 3ac$, $2bb - 6ac$, $abb - 3aac$, $2abb - 6aac$
1	

Compound formulas how refolved.

56. But if the laft quotient, or perhaps the formula itfelf at firft propofed, fhall ftill be compound, and yet is not divifible, after the foregoing manner, by any fimple quantity, fo that all it's divifors are compound terms; the way of obtaining them is different, and may be thus. The quantity is to be fet in order according to fome one of it's letters, as has been already fhown at § 24; and if there are fractions, they muft be reduced to a common denominator. Then all the divifors of the laft term muft be found, compounded of numeral divifors if there are any, and of the letter of one dimenfion. And if the greateft term has a numeral co-efficient, it muft be divided by fome one of thofe divifors, by which that co-efficient of the greateft term is divifible. By every one of thefe divifors, firft added and then fubtracted from the letter, by which the formula is ordered, the divifion muft be tried; and all thofe by which it fucceeds will be fo many divifors of the propofed quantity.

Let the formula $y^3 - 4ay^2 + 5a^2y - 2a^3$ be given. The divifors of one dimenfion of the laft term are a and $2a$. Therefore the divifion muft be tried by each of thefe added to the letter y, or fubtracted from it, becaufe the co-efficient of the greateft term y is unity; that is, by $y \pm a$, or by $y \pm 2a$. Firft let it be divided by $y - 2a$, and the quotient is $yy - 2ay + aa$, which alfo is divifible by $y - a$, giving $y - a$ in the quotient. Wherefore the divifors of the formula propofed are $y - a$, $y - a$, and $y - 2a$, from the product of which it is derived.

Let

Let the formula be $6y^4 - ay^3 - 21aayy + 3a^3y + 20a^4$. The diviſors of one dimenſion of the laſt term are a, $2a$, $4a$, $5a$, $10a$, $20a$; and becauſe the firſt term $6y^4$ is diviſible by 1, 2, and 3, we muſt try the diviſion by $y \pm \frac{1}{2}a$, $y \pm a$, $y \pm 2a$, $y \pm \frac{5}{2}a$, $y \pm 5a$, $y \pm 10a$, $y \pm \frac{1}{3}a$, $y \pm \frac{2}{3}a$, $y \pm \frac{4}{3}a$, $y \pm \frac{5}{3}a$, $y \pm \frac{10}{3}a$, $y \pm \frac{20}{3}a$. But becauſe it would be too tedious and troubleſome to try all theſe diviſors; in order to know among ſo many which are to be ſelected, we may make $y = z + a$; and ſubſtituting this in the place of y, and alſo it's powers, there will ariſe another formula, which is this.

$$\begin{array}{rrrrr}
6z^4 & + 24az^3 & + 36aazz & + 24a^3z & + 6a^4 \\
 & - az^3 & - 3aazz & - 3a^3z & - a^4 \\
 & & - 21aazz & - 42a^3z & - 21a^4 \\
 & & & + 3a^3z & + 3a^4 \\
 & & & & + 20a^4
\end{array}$$

Which by collecting the terms will be this,

$$6z^4 + 23az^3 + 12a^2z^2 - 18a^3z + 7a^4.$$

Now all the diviſors of the laſt term $7a^4$ of this formula are found to be a and $7a$, which divided by 2 and by 3, the numeral diviſors of $6z^4$, will make $\frac{1}{2}a$, $\frac{1}{3}a$, $\frac{7}{2}a$, $\frac{7}{3}a$. And becauſe it was made $y = z + a$, if theſe diviſors can be made uſe of in the ſecond given formula by z, they will alſo be uſeful in the firſt by y, when they are increaſed by the quantity a, that is by making them $\frac{3}{2}a$, $\frac{4}{3}a$, $\frac{9}{2}a$, $\frac{10}{3}a$. Therefore let theſe diviſors be compared with the diviſors of the firſt formula, and chooſe only thoſe which agree with them, that is $\frac{4}{3}a$ and $\frac{10}{3}a$, by which added to and ſubtracted from y, the diviſion muſt be tried; which will ſucceed with $y + \frac{4}{3}a$. But notwithſtanding this operation, if there ſhould ſtill remain too many diviſors to be ſelected by this compariſon, we may make $y = z - a$, and another formula will ariſe. From the diviſors found by this, the quantity a muſt be ſubtracted, and then they are to be compared with thoſe which are ſelected by means of the ſecond; and by them which agree, which will be fewer in number, the diviſion is to be tried. And proceeding in the ſame way of operation by new ſubſtitutions, making $y = z + 2a$, $y = z - 2a$, &c. the diviſors may be reduced to ſuch ſmaller numbers as will be ſufficient.

57. When

How the co-efficient of the firſt term may be removed.

57. When the propoſed formula has it's firſt or greateſt term multiplied by any number, inſtead of applying the rule aforegoing to this caſe, it may be more convenient to change the formula into another, the firſt term of which is multiplied only by unity; and then find the diviſors of the ſame, from which you may afterwards paſs to thoſe of the propoſed formula.

Let the formula be, for example,

$$\begin{array}{l} 3y^3 + 9ayy - 12aay - 12aab. \\ \quad + 3byy + 9aby \end{array}$$

Make $3y = z$, (or, in general, $ny = z$, putting n to repreſent the numeral co-efficient of the higheſt power,) and thence $y = \frac{1}{3}z$. This being ſubſtituted inſtead of y, and it's powers expreſſed in like manner, we ſhall have the formula $z^3 + 9az^2 + 3bz^2 - 36a^2z + 27abz - 108a^2b$, all divided by 9. Let the diviſors of this be found, (at preſent omitting the denominator 9,) which will be $z + 12a$, $z - 3a$, $z + 3b$; and taking account of the denominator 9, one of theſe is to be divided by 9, or two of them by 3, and they will be, for example, $z + 12a$, $\frac{z - 3a}{3}$, $\frac{z + 3b}{3}$; but it was made $3y = z$; and ſubſtituting this value of z in the diviſors, they will become $3y + 12a$. $y - a$, $y + b$, which are the three diviſors of the formula propoſed.

SECT. II.

Of Equations, and of Plane Determinate Problems.

Equations and their affections what?

58. *Equation* is a relation of equality, which two or more quantities, whether numerical, geometrical, or phyſical, have with one another when compared together; or which they have with nothing when compared to that. The aggregate of all thoſe terms which are wrote before the mark of equality, is called the Firſt Member of the Equation; and the aggregate of all thoſe which are wrote after it, is called the Second Member, or the *Homogeneum Comparationis*. Thoſe terms of the equation are homogeneous, when each of them is of the ſame dimenſion; and therefore in an equation they are ſaid to obſerve the law of homogeneity, as in this equation $axx - bbx = a^3$. And thus, on the contrary, they are ſaid not to obſerve the law of homogeneity, when the terms are not ſuch, as in this equation $x^4 - ax^2 = b$.

59. A

59. A *Problem* is a propoſition in which it is required to do or to find ſomething, by means of other things which are known, or of certain conditions which are given, and therefore called the *Data* of the Problem. So thoſe things which are required are the *Quæſita* of the Problem.

A problem, what.

60. Of Problems ſome are *Determinate*, and others *Indeterminate*. The determinate are thoſe which have a certain number of ſolutions, or which can be reſolved by one or more determinations, but always in a finite and limited number. Such it would be if we ſhould inquire, where we ought to cut the right line AB, ſo that the whole line, to it's greater ſegment, ſhould have the ſame ratio, as the greater ſegment to the leſſer. Becauſe one point only can be aſſigned in this line, for example C, which will have the property required. The ſame thing would be, if in a given circle AED we were to find a point, ſuppoſe C, in the diameter AD, from whence raiſing a perpendicular CE, terminated in the periphery; this perpendicular ſhould be juſt equal to a third part of the diameter. For there are only two points, each at an equal diſtance from the centre, that can ſatisfy this demand.

When problems are determinate, —when indeterminate.

Fig. 1.

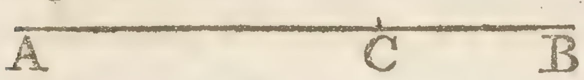
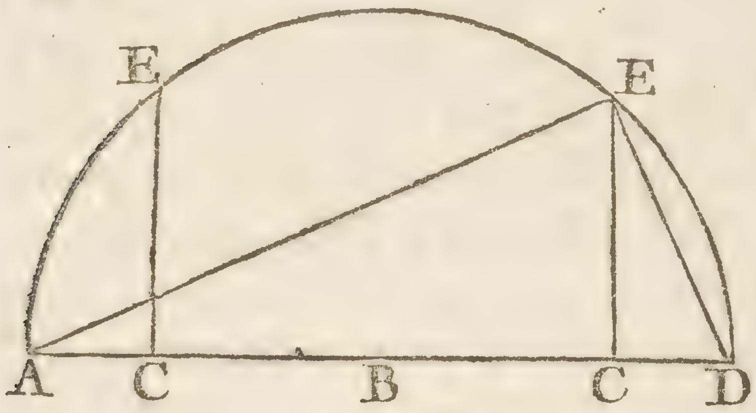


Fig. 2.



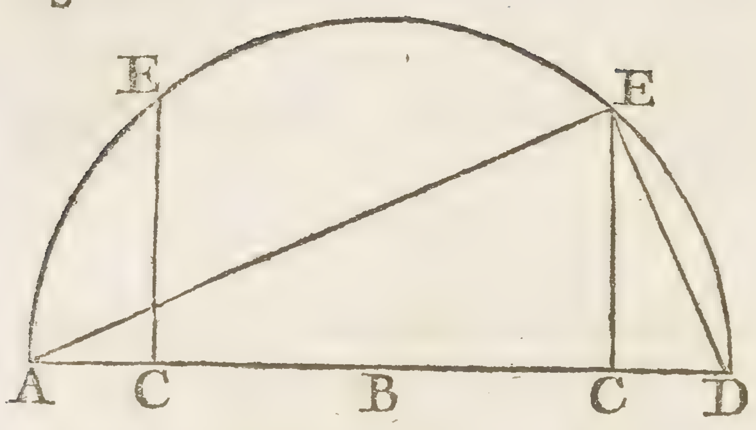
Now if it were propoſed to find, out of the right line AD, ſuch a point E, ſo that drawing from it two right lines EA, ED, to it's extremities A and D, the angle AED ſhall be a right angle; it will be found, that there are infinite ſuch points as will reſolve the problem, or the whole periphery AED, as is known from *Euclid*. In the ſame manner, if a point C is required in the diameter AD, from whence raiſing the perpendicular EC in the circle, it ſhall be a mean proportional between the ſegments AC, DC; it will be found, that all the points of the diameter will ſolve the problem (and therefore ſuch points are infinite in number); which is therefore called an *Indeterminate Problem*.

Determinate problems have occaſion for one unknown quantity only, but indeterminate ones of two at leaſt, though the manner of forming an equation is the ſame in both. Of theſe I ſhall treat particularly in Sect. III.

61. The given or known quantities are uſed to be denominated by the firſt letters of the alphabet, as has been ſaid already; but the unknown, or ſuch as are required, by ſome one of the laſt letters. And here it may be obſerved, that if the quantity ſought is a line, it ought always to have it's origin or beginning at ſome determinate fixed point. And as that which is required is already ſuppoſed to be done or known, by calling it, for example, x; ſo that from theſe quantities ſuppoſed as known, others that depend on them come to

Known and unknown quantities how diſtinguiſhed.

Fig. 2.



be known or given, as it were by hypothesis. Thus, AD $= a$ being given, and C being supposed the point required, and therefore calling AC x, it will be CD $= a - x$; and thus we may argue of several others. And further, though many of the quantities are not expressly given, like as in the line AD; yet, however, they are given implicitly, and as it were by construction. Thus, in the right-angled triangle AED, if the hypothenuse AD $= a$ is given, and the side ED $= b$; then, by the 47th proposition of the first Book of *Euclid*, the side AE $= \sqrt{aa - bb}$ will be therefore given. Thus, in the semicircle AED, the diameter AD $= a$ being given, and the segment AC $= b$, it will be CD $= a - b$; and therefore, by *Euclid*, vi. 8, it will be CE $= \sqrt{ab - bb}$. Or because AC was called x, it will be CE $= \sqrt{ax - xx}$, which is given both by hypothesis and by construction. Thus, in the right-angled triangle ACB, from the right angle B letting fall the perpendicular BD, let be given, for example, the two lines AC $= a$, and AB $= b$; then in like manner will be given all the other lines BC, BD, AD, DC. For BC $= \sqrt{aa - bb}$, by *Euclid*, i. 47, as said before. And by vi. 8, CD will be a third proportional to AC and CB; wherefore it will be CD $= \frac{aa-bb}{a}$, by the 17th of the same book. AD will be a third proportional to AC and AB, and therefore AD $= \frac{bb}{a}$. DB will be a mean proportional between AD and DC; or else it will be a fourth proportional to AC, CB, AB, and therefore, by 16 of the same book, it will be DB $= \frac{b\sqrt{aa-bb}}{a}$. Thus, in the right-angled triangle ABC, if DH is parallel to BC, and are given AB $= a$, BC $= b$, AD $= x$; then, by 4 of vi., will be given DH $= \frac{bx}{a}$, AH $= \frac{x\sqrt{aa+bb}}{a}$. And the same may be observed of infinite others.

Fig. 3.

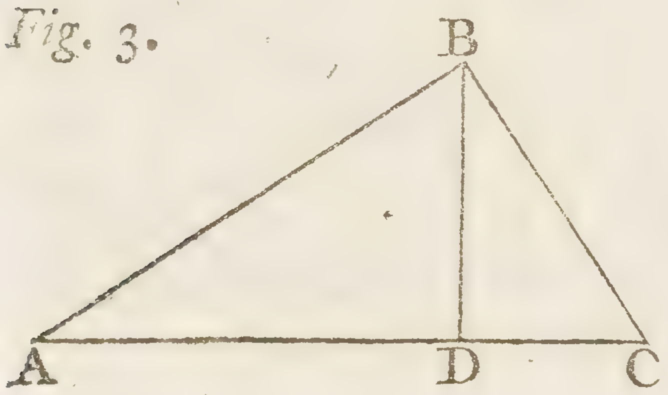
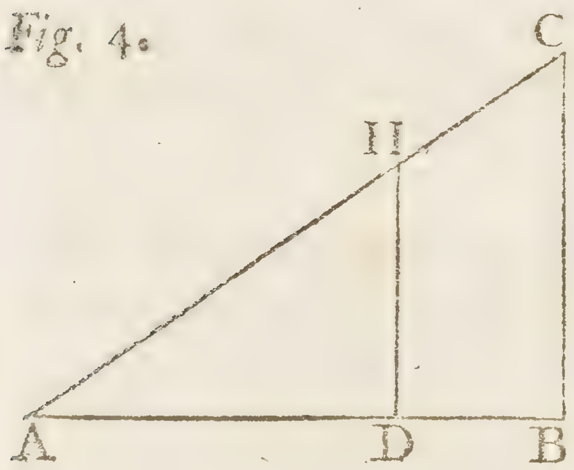


Fig. 4.



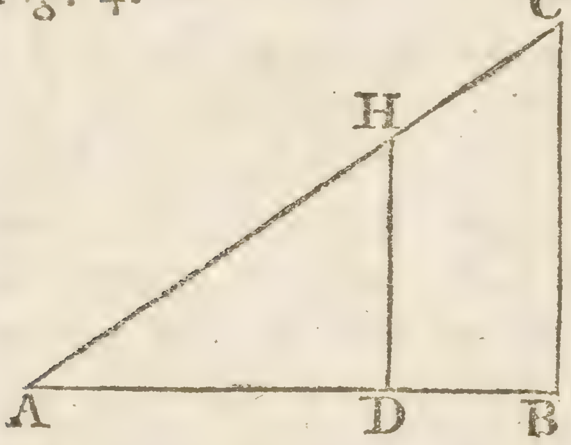
Equations how derived.

62. Thus, by supposing that already done or known, which is to be done or known, and by treating given and sought quantities indifferently, all the conditions may be fulfilled, which are required by the proposition or problem, and we shall thus arrive at an equation. Let there be a right line AB, which is to be cut in extreme and mean proportion. Let AB $= a$, and let C be the point

Fig. 1.

point required. Let AC $= x$, and therefore CB $= a - x$. The condition implied is, that it ought to be AB . AC :: AC . CB; that is, $a . x :: x . a - x$. But by the nature of a geometrical proportion, the rectangle of the means muſt be equal to that of the extremes; ſo that $aa - ax = xx$, and thus we are now come to an equation. Again, let there be three numbers given, the firſt is 4, the ſecond is 5, and the third is 10. A fourth number muſt be found, ſuch that, if from the product of this into the third the firſt be ſubtracted, and if the remainder is divided by the firſt, the quotient ſhall be equal to the ſecond number given. Let the number ſought be denoted by x; then the product of this into the third will be $10x$, from which ſubtracting the firſt, the remainder will be $10x - 4$, and dividing this by the firſt, the quotient will be $\frac{10x - 4}{4}$, which by the condition of the problem ſhould be 5, that is $\frac{10x - 4}{4} = 5$, which is the equation required.

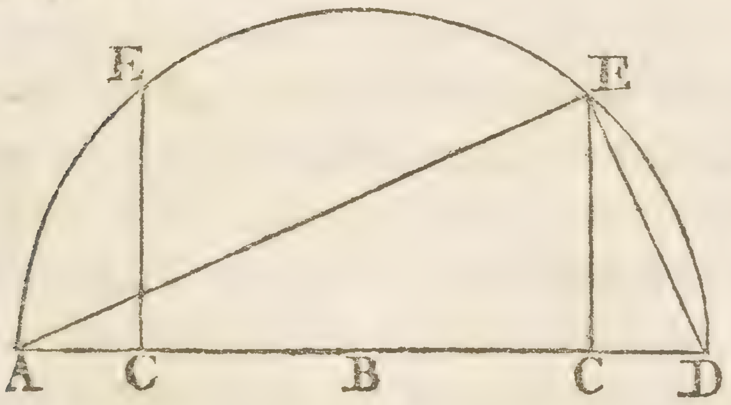
Fig. 4.



Again, in the triangle ABC, are given the ſides AC $= a$, BC $= b$, and the baſe AB $= c$; we are to find in this ſuch a point D, that drawing DH parallel to BC, the ſquare of DH may be equal to the rectangle AD $\times$ DB. Make AD $= x$, whence DB $= c - x$; and becauſe of like triangles ABC, ADH, it will be DH $= \frac{bx}{c}$. Then by completing what the problem requires, we ſhall have the equation $\frac{b^2x^2}{c^2} = cx - xx$.

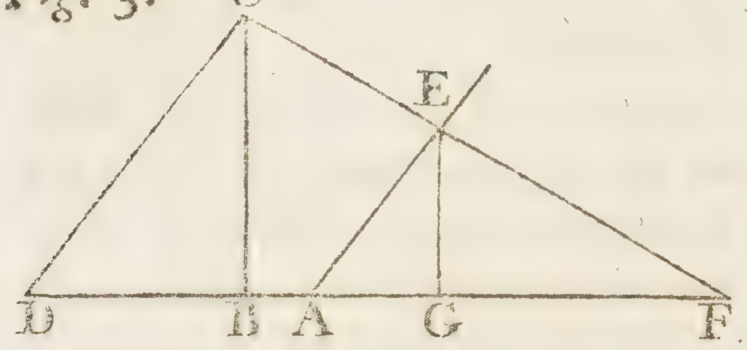
63. If the given triangle ABC is right-angled at B, we ſhall have no need to denominate AC $= a$, but otherwiſe $= \sqrt{bb + cc}$, to expreſs thereby the condition of a right-angled triangle. Thus in the ſemicircle AED is given the diameter AD $= 2a$, and the ſegment AC $= b$; hence conſequently is given the line CE, and therefore it ought not to be expreſſed by a letter at pleaſure, but to be denominated from the property of the circle, by making it $= \sqrt{2ab - bb}$; thereby expreſsly to indicate, that it is an ordinate in the circle at the point C. And in general it is to be underſtood, that the ſame ought to be done in all like caſes. Some lines to be denominated by inference.

Fig. 2.



64. But perhaps it may make ſome difficulty, that very often the lines given in a figure, by which the problem is propoſed, are not ſufficient to obtain ſuch quantities or denominations, as are neceſſary to arrive at an equation. Such a New lines to be drawn.

Fig. 5.



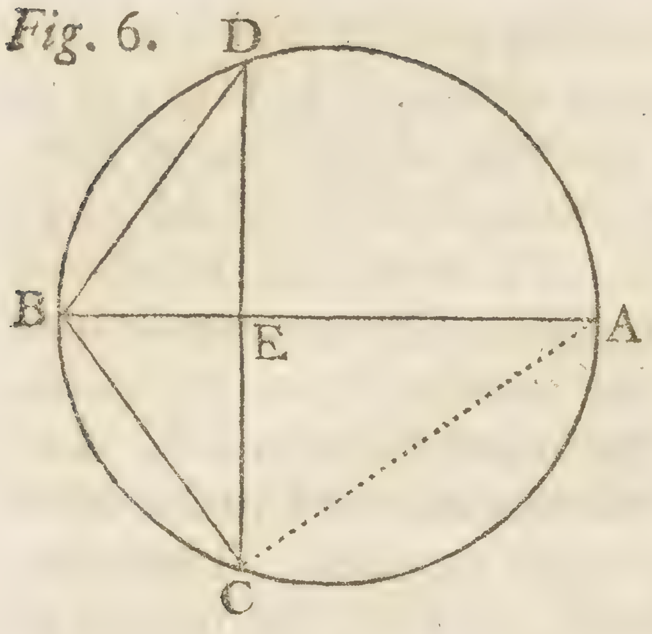
case would be, if two indefinite right lines AE, AF, were given in position, and a point C out of those lines: and if it were proposed to draw a line CF in such a manner from the point C, as that it should include a triangle AEF, equal to a given plane. The expression of the triangle AEF would be half the rectangle of AF into EG, letting fall EG perpendicular to AF. Now make AF $= x$; but yet it will not be possible to determine the value of EG from the lines hitherto described. Upon such occasions it will be necessary to construct or complete the figure, by drawing parallels, raising or letting fall perpendiculars, forming similar triangles, describing circles, or by using the like expedients of the common Geometry; for which it is not possible to give any general rules, as they will depend on the various circumstances of problems, on sagacity, industry, and practice, and often upon chance. But commonly these propositions of the first Book of *Euclid* are used to be of good service, 5, 13, 15, 27, 29, 32, 47; some of the second; these of the third, 20, 21, 22, 27, 31, 35, 36; these of the sixth, 1, 2, 3, 4, 5, 6, 7, 8; and some of the 11th and 12th when solids are concerned. Therefore, in the problem now proposed, from the point C draw CD parallel to EA, and EG, CB, perpendicular to FA produced. Now because the right lines AE, AF, are given in position, and also the point C; the lines AD, CB, will be given in magnitude. Therefore make AD $= a$, CB $= b$, AF $= x$, and let the given plane be $= cc$. And as the triangles FDC, FAE, are similar, as also the triangles DCB, AEG; we shall have the analogies DF . AF :: (DC . AE ::) BC . EG. That is, $a + x \,.\, x :: b \,.\,$ EG. Therefore EG $= \frac{bx}{a+x}$. And because the triangle AEF, that is, half the rectangle of AF into EG, ought to be equal to the given plane cc, we shall at last have the equation $\frac{bxx}{2a+2x} = cc$.

Equations how formed from different values of the same quantity.

65. The proposing of the problems only, which hitherto I have taken for examples, has brought me immediately and directly to an equation; because it was required that the two quantities so found should be made equal. But this method will not thus succeed, when from certain quantities given, it shall be proposed to find others, without such a condition as will lead us expresly to an equation. Then it may be needful to use a little art to obtain it, and that will be by means of different properties, and compounding the figure if necessary, to find two different expressions of the same quantity, and so to make an equation between them. I said by means of different properties, because the same property, however managed, will always give the same expression. I shall produce three examples of this, which I think may suffice at present.

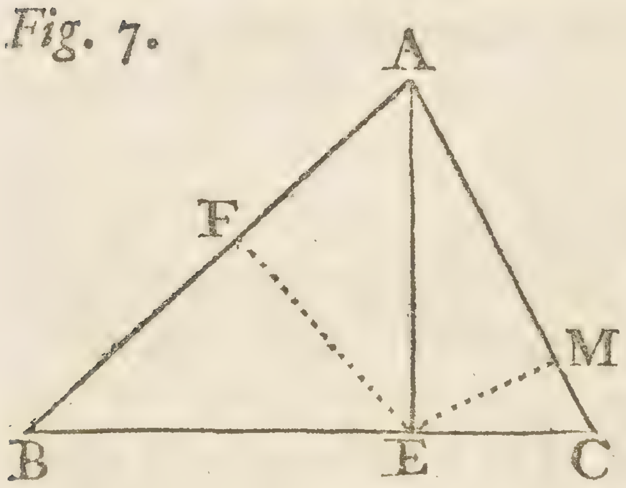
Given

Fig. 6.



Given the isosceles triangle CDB, the diameter AB of the circle CADB is required, in which it may be inscribed. Make CD $= a$, CB $=$ BD $= b$, BA $= x$, which is the diameter required; and draw CA. The two triangles ABC, BCE, will be similar, because the angles BCA and CEB are right ones, and the angle BCE $=$ BDC $=$ BAC. Therefore it will be AB . BC :: BC . BE; that is, $x . b :: b .$ BE; whence BE $= \frac{bb}{x}$. Moreover CE is the half of CD, whence CE $= \frac{1}{2}a$. And because of the right-angle CEB, it will be CB$q = \frac{aa}{4} + \frac{b^4}{xx}$. But the square of CB is also $= bb$. Therefore we shall have the equation $bb = \frac{aa}{4} + \frac{b^4}{xx}$.

Fig. 7.

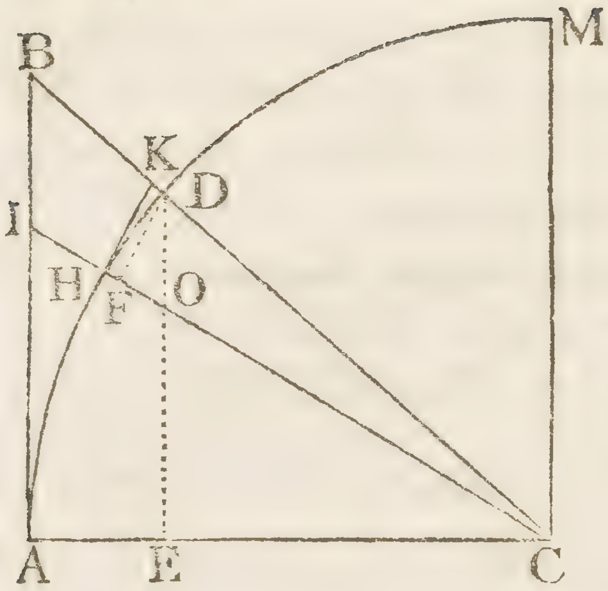


In the triangle ABC the three sides are given, and from the angle A letting fall the perpendicular AE upon BC; the two segments BE, EC are required. Make AB $= a$, AC $= b$, BC $= c$, BE $= x$; then it is EC $= c - x$. By the 47 of the first of *Euclid*, the square of AE will be equal to the square of AB, subtracting the square of BE; that is AE$q =$ AB$q -$ BEq. But by the same it will be also AE$q =$ AC$q -$ ECq. Therefore AB$q -$ BE$q =$ AC$q -$ ECq. And reducing to an algebräick expression, it will be $aa - xx = bb - cc + 2cx - xx$, that is, $aa = bb - cc + 2cx$.

Again another way. Let EF be drawn perpendicular to AB; then, by the 8 of the sixth of *Euclid*, it will be AB . BE :: BE . BF; and therefore BF $= \frac{xx}{a}$. Thence AF $= a - \frac{xx}{a}$. And, by the same proposition, it will be AF . AE :: AE . AB; and therefore AE$q = aa - xx$. From the point E drawing the right line EM perpendicular to AC, by the same way of arguing it will be found, that AE$q = bb - cc + 2cx - xx$; and making a comparison between these two values, we shall have the same equation as before.

The

Fig. 8.



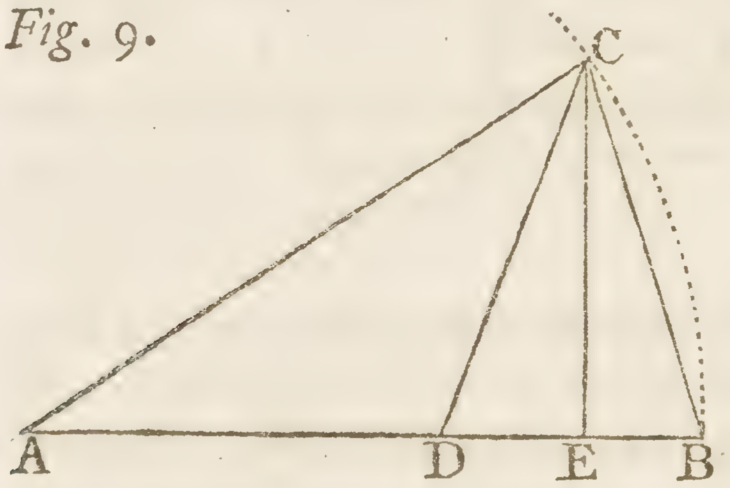
The quadrant AHM being given, and the tangents AI, HK, of the two arches AH, HD; it is required to find AB the tangent of the ſum of theſe two arches. Make the radius CA $= a$, AI $= b$, HK $= c$, and AB $= x$. To obtain an equation, from the point D let be drawn DE perpendicular upon AC. Then by the ſimilar triangles CBA, CDE, we may find the values of CE and DE. Let us examine then if we cannot contrive to denominate the ſame DE in another manner. Therefore drawing DF perpendicular to CH, by means of the ſimilar triangles CAI, CEO, we may have the lines EO, CO; and in like manner, by means of the ſimilar triangles CHK, CFD, we may have the line FD; and from the ſimilar triangles CEO, FOD, we may obtain OD; whence we ſhall finally arrive at OD in another manner, independent on the firſt, and then ED $=$ EO $+$ OD, which will give us an analytical equation.

I have here produced the order of arguing only, which might be uſed to bring us to an equation; omitting the actual operation, becauſe the problem will be completely ſolved in another place.

How we are to proceed when angles are concerned.

66. It will often require ſome particular expedients to be made uſe of, in ſuch problems in which angles are concerned; for by ſome artifice we muſt paſs from the properties of angles to thoſe of lines, which may enter the problem in their ſtead. I will take an example of this from the 10th propoſition of the fourth Book of *Euclid*. Let it be required, upon the given right line AB, to conſtruct an iſoſceles triangle ABC, of which the angle at A ſhall be half of either of the angles ABC, or ACB. Let the triangle ABC be ſuch a triangle, and therefore the two angles ACB, ABC, will be equal to each other, and thence the ſides AC, AB, will alſo be equal. Let the right line CD be drawn in ſuch a manner, that it may biſect the angle ACB. Then the two triangles ACB, CDB, will be ſimilar, from whence we ſhall have this analogy, AB . BC :: BC . BD. But it is BC $=$ DC $=$ AD, and therefore it will be AB . AD :: AD . DB. And now ſee the problem propoſed reduced to another, which is, to divide the given line AB in extreme and mean proportion. Wherefore this ſecond problem being reſolved, the point D will be found, and the problem at firſt propoſed will then be ſolved. For biſecting DB in E, and raiſing the perpendicular EC, it will meet in C an arch BC, which is deſcribed with radius AB from the centre A. Then if from the point found C we draw the lines CA, CB, the triangle ACB ſhall be ſuch as is required.

Fig. 9.



67. Now

67. Now when the equation of a problem is found, all that remains to be done is, to derive the value of the unknown quantity from it; that is, to reduce the unknown quantity to be equal to fome known and given quantities, in which confifts the folution of the problem. And this is called the *Refolution of the Equation.* Equations how reduced.

For this purpofe we muft call to our affiftance the following Axioms.

1. If to two equal things we fhall add equals, or if we fhall fubtract equals from them, the fums or the remainders will alfo be equal.

2. If equal things are multiplied or divided by equals, the products or quotients will alfo be equal.

3. If from equals a root be extracted with an equal index, the roots or quantities refulting will be equal.

4. If equals are raifed to a power with an equal index, thofe powers or refulting quantities will be equal.

From the firft of thefe axioms we learn, that if we fhould defire that any term of an equation, which is on one fide of the mark of equality, fhould pafs to the other fide; this may always be done without deftroying the equality of the terms. Let the equation be $ax + bb = -xx + cc$; if we add xx to both the members of this equation, it will be $ax + bb + xx = xx - xx + cc$, in which $xx - xx$ expunge one another, and there will remain $ax + bb + xx = cc$, where the term xx has paffed into the firft member of the equation; from whence if bb is to be taken away, it will be $ax + bb + xx - bb = cc - bb$; but $bb - bb$ expunging one another, the remaining equation will be $ax + xx = cc - bb$, where the term bb has paffed into the fecond member of the equation. Wherefore in general, when we would have any term pafs from one fide of the equation to the other, it will be enough to expunge it on one fide, and write it on the other with it's fign changed. In confequence of this, we may at pleafure make a term pofitive which in the equation is negative, and fo on the contrary; and that will be by writing it on the oppofite fide, and changing it's fign. Therefore $aa - xx = bb$ will be the fame as $aa - bb = xx$, or $xx = aa - bb$. Wherefore if there fhall be the fame term on each fide of the equation, and affected with the fame fign, they may both be expunged without injuring the equation. As, if it were $ax - xx = bb - xx$, it would be reduced to $ax = bb$. For, tranfpofing the term $- xx$, it would be then $ax + xx - xx = bb$, where $xx - xx$ deftroy each other. The fame thing would follow, if, inftead of tranfpofing the term which is common to both members, it were added to both if in the equation it were negative, or fubtracted from both if affirmative.

68. From the fecond axiom we learn, that if an equation fhould have fractions in it, it may always be freed from them without prejudice to the equation; by reducing every term to a common denominator, and then rejecting that deno- Reduced by multiplication.

denominator: becaufe equal quantities multiplied by equals make equal products. Let the equation be $a - \frac{xx}{b} = b$. Reducing all to a common denominator, it will be $\frac{ab - xx}{b} = \frac{bb}{b}$, and multiplying all by b, or rejecting the common denominator, it will be $ab - xx = bb$. And if befides we would have the term $- xx$ to be pofitive, it will be $ab = bb + xx$, or otherwife $xx = ab - bb$. Let the equation be $\frac{ax}{2} - \frac{bxx}{a} = ab$. Reducing to a common denominator, it will be $\frac{aax - 2bxx}{2a} = \frac{2aab}{2a}$, and multiplying all by $2a$, it will be $aax - 2bxx = 2aab$. And if we defire befides, that the term $- 2bxx$ fhould be pofitive, and moreover that all the terms in which the letter x is concerned fhould be on one fide of the equation, make $2bxx - aax = - 2aab$; or reducing the whole equation to one fide, by which it will be equal to nothing, it will be $2bxx - aax + 2aab = 0$.

Reduced by divifion.

69. By the fame axiom we may free any letter, or any power of a letter, in any equation, from it's co-efficient, or from any quantity in which it happens to be multiplied; and that is by dividing every term by that co-efficient. Now let there be $2bxx - aax = - 2aab$, and let it be required to free the term $2bxx$ from it's co-efficient $2b$. Then dividing each member of the equation by the fame quantity $2b$, the quotients $\frac{2bxx - aax}{2b} = - \frac{2aab}{2b}$ fhall ftill be equal, and therefore $xx - \frac{aax}{2b} = - aa$. Again, if the equation is $ax - \frac{a^3}{b} = bb - \frac{3bxx}{2a} - bx$, and if it were defired that xx fhould be pofitive, freed from it's fraction and co-efficient, and that all the terms which any how contain the letter x fhould be on one fide of the equation, and known terms on the other; write then $\frac{3bxx}{2a} + bx + ax = bb + \frac{a^3}{b}$, multiply all the terms by $2a$, and it will be $3bxx + 2abx + 2aax = 2abb + \frac{2a^4}{b}$; then divide every term by $3b$, and the equation will become $xx + \frac{2}{3}ax + \frac{2aax}{3b} = \frac{2}{3}ab + \frac{2a^4}{3bb}$, which has all the conditions required.

Reduced by raifing powers.

70. From the fourth axiom we learn, that if an equation contains radicals or furds, it may be freed from them, by writing the furd term or terms on one fide of the equation, and the rational quantities on the other, and then fquaring each member of the equation if the root is quadratick, or cubing if cubick, &c. Thus if we had $\sqrt{aa - xx} + a = x$, we muft write it thus, $\sqrt{aa - xx} = x - a$, and then fquaring, $aa - xx = xx - 2ax + aa$, that is $2ax = 2xx$, or $x = a$.

Thus

Thus if the equation were $\sqrt[3]{aax - x^3} - a + x = 0$, write it $\sqrt[3]{aax - x^3} = a - x$, and it will be, by cubing, $aax - x^3 = a^3 - 3a^2x + 3ax^2 - x^3$. That is $4a^2x - 3ax^2 - a^3 = 0$, or by dividing by a, $4ax - 3x^2 - a^2 = 0$.

But if the radical terms be two or more, ſo that they will not vaniſh at one operation, it muſt be repeated as often as there is occaſion. Thus $\sqrt{bx} = a + \sqrt{ax}$: write it thus, $\sqrt{bx} - \sqrt{ax} = a$; then ſquaring, it is $bx - 2\sqrt{abxx} + ax = aa$, that is $bx + ax - aa = 2\sqrt{abxx}$. And ſquaring again, $bbxx + aaxx + a^4 + 2abxx - 2aabx - 2a^3x = 4abxx$; that is $b^2x^2 - 2abx^2 + a^2x^2 - 2a^2bx - 2a^3x + a^4 = 0$. Thus $y = \sqrt{ay + yy - a\sqrt{ay - yy}}$ by ſquaring will be $yy = ay + yy - a\sqrt{ay - yy}$, that is $ay = a\sqrt{ay - yy}$, or $y = \sqrt{ay - yy}$. And ſquaring again, $yy = ay - yy$, or $2y = a$.

71. Theſe things being premiſed, the manner of reſolving equations will be eaſy, in order to obtain the value of the unknown quantity, in ſuch terms as are known and given, and which ſerve to the ſolution of the problem. But firſt the equations are ſuppoſed to be freed from all aſymmetry, that is from radicals, if the unknown quantity be under a *vinculum*; and then reduced to the moſt ſimple expreſſion; by expunging ſuperfluous terms, if ſuch there be; by dividing of each member that ſhall be multiplied by the ſame quantity; or by multiplying if ſo divided. As if, for example, we had $\frac{axx - aax + aab}{b} = \frac{a^3 + aab}{b}$, it would be reduced to $xx - ax = aa$. Further, by the firſt term of an equation is meant the aggregate of all thoſe terms, which contain the higheſt power of the unknown quantity. By the ſecond term is meant the aggregate of all thoſe terms which contain the next inferior degree of the ſame quantity, and ſo on. By the known term is meant, the aggregate of all thoſe terms which do not at all contain the unknown quantity. Whence in the equation $axx - bxx - bbx - aax = a^3 - b^3$, or elſe $axx - bxx - bbx - aax - a^3 + b^3 = 0$, the firſt term will be $axx - bxx$, or $\overline{a - b} \times xx$. The ſecond will be $- bbx - aax$, that is $- \overline{aa + bb} \times x$. The known term is $- a^3 + b^3$. In the equation $aaxx - abxx + a^4 - b^4 - a^3b = 0$, the firſt term will be $\overline{aa - ab} \times xx$; the ſecond is wanting, and the known term is $a^4 - b^4 - a^3b$. In the equation $ax^3 + bx^3 - aaxx - a^4 = 0$, the firſt term will be $\overline{a + b} \times x^3$, the ſecond $- a^2x^2$, the third is wanting, and the fourth or known term is $- a^4$. And thus it is to be underſtood in all other equations. Here it may be obſerved, that a term ſuch as $aaxx - bbxx$, (which is likewiſe to be underſtood of any other compound term, having contrary ſigns,) may be either a poſitive or negative quantity; it will be poſitive if a be greater than b, but negative if the contrary. So that when it ſhall be ordered hereafter to make ſuch a term of an equation poſitive, we muſt have regard to this explanation.

How equations are to be reſolved.

Equations further reſolved, and firſt ſimple ones.

72. This being ſuppoſed, in order to reſolve an equation; firſt, if it have a fraction, in the denominator of which the unknown quantity is found, it muſt be reduced to a common denominator. Secondly, the term of the higheſt power of the unknown quantity muſt be made poſitive, and all the terms containing the unknown quantity muſt be wrote in order on one ſide of the equation, and the known terms on the other ſide. And thirdly, if the firſt term, or that which contains the higheſt power of the unknown quantity, ſhould have a denominator, it muſt be freed from it's fraction by what is ſaid, § 68. Laſtly, if it have a co-efficient, or be multiplied into any given quantity, it muſt be freed from this, by what has been taught, § 69.

Hence it is eaſy to perceive, that by proceeding after this manner, if the equation ſhall be ſimple, or have an unknown quantity of one dimenſion only, it will be now intirely reſolved, and that unknown quantity will be found equal to known quantities only, which was the thing propoſed to be done. As if the equation were $aa - ff = \frac{bbx - aax}{2m}$, and aa were greater than bb. Then to make that term poſitive which contains the unknown quantity, write it thus, $\frac{aax - bbx}{2m} = ff - aa$; and freeing it from the denominator, it will be $aax - bbx = 2mff - 2maa$; and then from the co-efficient, it will be $x = \frac{2mff - 2maa}{aa - bb}$, in which the value of x is now intirely known. If aa were leſs than bb, we might then write it thus, $x = \frac{2maa - 2mff}{bb - aa}$, which comes to the ſame without any occaſion of tranſpoſition.

Equations reſolved, having ſimple powers.

73. When the unknown quantity is raiſed to any power, which power is the ſame in all the terms in which it is found; or, which is the ſame thing, if all thoſe terms are conceived to make but one term; then the equation is to be reſolved by the third axiom before, and we ſhall have the unknown quantity equal to known quantities only, by extracting ſuch a root out of both members of the equation, as is denoted by the index of that power. Let the equation be $bb = aa - \frac{axx + bxx}{2c}$. Now to make the term poſitive in which x is found, write $\frac{axx + bxx}{2c} = aa - bb$; and to free it from it's fraction and co-efficient, write it $xx = \frac{2c \times \overline{aa - bb}}{a + b}$, or by diviſion, $xx = 2c \times \overline{a - b}$; and laſtly, by extracting the ſquare-root, $x = \pm \sqrt{2ac - 2bc}$. Here I put the ſign of the root ambiguous, becauſe of what is ſaid at § 15. For the ſame reaſon, if it were $x^3 = a^3 + b^3$, we ſhould have $x = \sqrt[3]{a^3 + b^3}$; and ſo of all others in general.

74. But

74. But if the equation contain the unknown quantity raiſed to it's ſquare, together with the rectangle or product of the ſame into known quantities, which is called the ſecond term (and ſuch an equation is called an *Affected Quadratick*, as it is called a *Simple Quadratick* when this ſecond term is wanting); this being prepared as is aforeſaid, to both members of the equation muſt be added the ſquare of half the co-efficient of the ſecond term, (that is to ſay, the ſquare of half that quantity, whether integer or fraction, by which the unknown quantity is multiplied,) and then it is plain that the firſt member will always be a ſquare, the root of which will be the aggregate of the unknown quantity, and of the half co-efficient with it's proper ſign. And then extracting the root, this aggregate ſhall be equal to the ſquare-root of the other member of the equation; and tranſpoſing the half co-efficient as a known quantity, we ſhall finally have the unknown quantity equal to the ſum or difference (according to the nature of the ſigns) of the radical and the ſaid half co-efficient. Thus let the equation be $xx + 2ax = bb$: if we add to each member the ſquare of half the co-efficient of the ſecond term, that is aa, the equation will be $xx + 2ax + aa = aa + bb$, and extracting the ſquare-root, it will be $x + a = \pm\sqrt{aa + bb}$, and by tranſpoſing, it is $x = \pm\sqrt{aa + bb} - a$. Affected quadraticks reſolved.

Let the equation be $bbx - aax - mxx + \frac{aabb}{m} = 0$. Making the greateſt term poſitive, and ordering the equation, it will be $mxx + aax - bbx = \frac{aabb}{m}$, and dividing by m, and adding on both ſides the ſquare of half the co-efficient of the ſecond term, it will be $xx + \frac{aa - bb}{m}x + \frac{a^4 - 2aabb + b^4}{4mm} = \frac{a^4 - 2aabb + b^4}{4mm} + \frac{a^2b^2}{m^2}$; and extracting the ſquare-root, it is $x + \frac{aa - bb}{2m} = \pm\sqrt{\frac{a^4 - 2a^2b^2 + b^4}{4m^2} + \frac{a^2b^2}{m^2}}$, and reducing the radical to a common denominator, and tranſpoſing the known term $\frac{aa - bb}{2m}$, it will be $x = \frac{bb - aa}{2m} \pm \sqrt{\frac{a^4 + 2a^2b^2 + b^4}{4m^2}}$. But the root of this radical may be actually extracted, and is either $+\frac{aa + bb}{2m}$, or $-\frac{aa + bb}{2m}$, becauſe of the ambiguous ſign $\pm$. Therefore there will be two values of x, one is $x = \frac{bb - aa}{2m} + \frac{aa + bb}{2m} = \frac{bb}{m}$, and the other is $x = \frac{bb - aa}{2m} - \frac{aa + bb}{2m} = -\frac{aa}{m}$.

75. Therefore the ambiguity of the ſign, which the extraction of the ſquare-root always brings with it, ſupplies two values of the unknown quantity, which may be both poſitive, or both negative, or one poſitive and the other negative; and ſometimes both imaginary, according to the known quantities of which they are compoſed. For example, in the final equation $x = \pm\sqrt{aa + bb} - a$, one value or $\sqrt{aa + bb} - a$ will be poſitive, becauſe, as $\sqrt{aa + bb}$ is greater than a, The uſe of the ambiguous ſign.

the difference will be poſitive. The other value $-\sqrt{aa+bb}-a$ will be negative, as is evident. In the equation $x = a \pm \sqrt{aa-bb}$, (ſuppoſing b to be leſs than a,) both the values will be poſitive, becauſe $\sqrt{aa-bb}$ is leſs than a. And for the ſame reaſon, in the equation $x = \pm\sqrt{aa-bb}-a$, both the roots will be negative. Now, if b were greater than a, both would be imaginary, as I have already obſerved at § 15, becauſe then $\sqrt{aa-bb}$ would be the ſquare-root of a negative quantity. In the equation $x^4 = a^4 - b^4$, which requires twice the extraction of the ſquare-root, that is, $xx = \pm\sqrt{a^4-b^4}$, and thence $x = \pm\sqrt{\pm\sqrt{a^4-b^4}}$, there are four values of x; two real ones, of which one is poſitive and the other negative, that is, $x = \pm\sqrt{+\sqrt{a^4-b^4}}$, ſuppoſing b to be leſs than a; the other two are imaginary, that is, $x = \pm\sqrt{-\sqrt{a^4-b^4}}$; and when b is greater than a, all the four roots will be imaginary: and theſe obſervations may eaſily be applied to all other equations. Theſe negative values or roots, which by ſome authors are called falſe ones, are not leſs real than the poſitive, and have only this difference, that if, in the ſolution of a problem, the poſitive be taken from a fixed point, or beginning of the unknown quantity towards one part, the negative are taken from the ſame point towards the contrary part. Let A be the beginning of the unknown quantity x in a certain problem, and let the final equation (for example) be $x = \pm a$. If we take AB $= a$, and it be determined that the poſitive values ſhall proceed from A towards B; then ſhall AB $= a$ be the poſitive value of x. And conſequently, taking AC $=$ AB, but on the contrary part from the point A, we ſhall have AC $= -a$, or the negative value of x. And the problem ſhall have two ſolutions, one at the point B, and the other at the point C. But the practice of all this will be beſt underſtood by the ſolution of the problems which are here to follow.

Fig. 10.

Uſe of imaginary quantities.

76. Therefore, whenever the equation to which we are led by the conditions of the problem ſhall ſupply us with none but imaginary values, this plainly declares, that the problem as now propoſed does not admit of a real ſolution, but is abſolutely impoſſible. The ſame thing is to be concluded, when the final equation brings us to an abſurdity, ſuch as if it ſhould give us a finite quantity equal to nothing, or the whole equal to the part, or ſuch like. We ſhould come to an abſurdity of this kind, if in the right line AB $= a$, it were propoſed to find ſuch a point C, as that the ſquare of the whole line ſhould be equal to the two ſquares of the two ſegments. For, making AC $= x$, it would be $aa = xx + \overline{a-x}|^2 = xx + aa - 2ax + xx$, that is $2xx = 2ax$, or $x = a$; which is as much as to ſay, that the part is equal to the whole. We ſhould likewiſe fall into an inconſiſtency, if, aſſuming

Fig. 1.

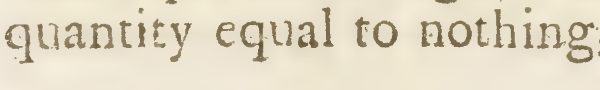
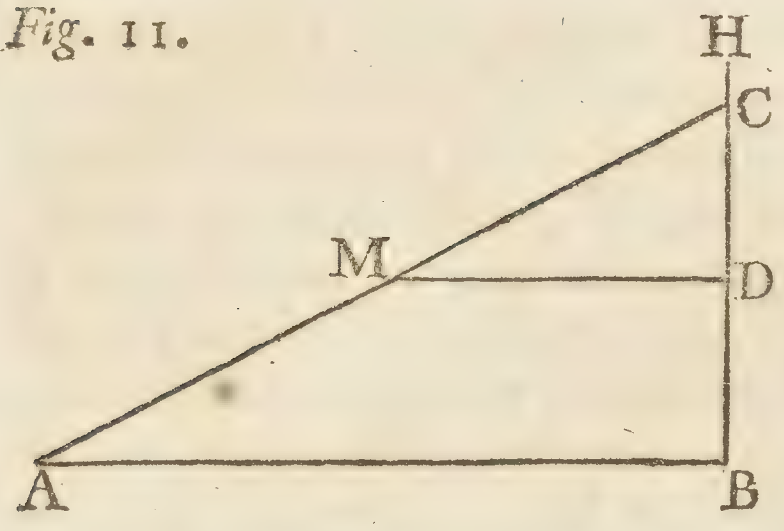


Fig. 11.



aſſuming a right line, as AB, and raiſing an indefinite perpendicular upon it BH, we ſhould ſeek for a point in this, as C, from whence we might draw the right line CA to the given point A, ſo as that the two lines CB, CA, may be parallel. For, making BA $= a$, BC $= x$, and taking BD $= \frac{1}{2}x$, and drawing DM parallel to BA; becauſe of ſimilar triangles CBA, CDM, it would be DM $= \frac{1}{2}a$. But if CA and CB are parallel, it ought to be DM $=$ BA, and therefore $\frac{1}{2}a = a$, which is an impoſſible equation.

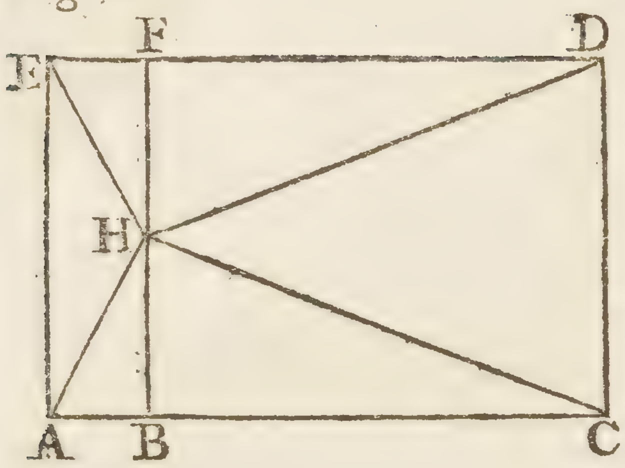
Now if it ſhould be pretended, that the firſt of the two foregoing equations, or $2xx - 2ax = 0$, is no otherwiſe abſurd, but that it ſupplies us with two values of x, which, though uſeleſs, are however real and conſiſtent; relying upon this argument, that if we divide the equation by $2x - 2a$, there will reſult $x = 0$, a real value which ſolves the problem. For taking $x = 0$, or dividing the line AB in the point A, one part of it will be 0, and the other will be a. Therefore the ſquare of the whole line will be equal to the ſquares of the two ſegments; that is, $aa = 0 + aa$. Now dividing the ſame equation by $2x$, there will reſult $x = a$, which is a real value, and reſolves the problem, by dividing the line in the point B. Whoever ſhould argue thus, as I ſaid before, I ſhould not venture to oppoſe him; but whatever is the true notion of this and ſuch like equations, it is however certain, that they only make us know what we knew before.

For an example of an equation which brings us to an abſurd concluſion, I have taken one which gives us a finite quantity equal to nothing, or the whole equal to the part. Yet this muſt be underſtood only when the unknown quantity cannot be of an infinite magnitude, and the problem is no more than a determinate problem; for otherwiſe ſuch equations may be very true, as will be ſeen hereafter.

77. Sometimes we may meet with equations which contain the ſame quantities on both ſides the mark of equality, and therefore when reduced bring us finally to this concluſion, that $0 = 0$. Such equations as theſe (which are called *Identical Equations*) inform us only, that the value of the quantity required may be what we pleaſe, as it vaniſhes out of the equation; and that the propoſition is rather a theorem than a problem. Here follows an example of this. What we learn from identical equations.

In

Fig. 12.



In the given rectangle ACDE, from a given point B in the fide AC is drawn BF parallel to the fide AE; in BF is required fuch a point H, that drawing the lines HA, HC, HD, HE, to the feveral oppofite angles, the fum of the fquares of HA, HD, fhall be equal to the fum of the fquares of HE, HC. Make $AB = a$, $BC = b$, $CD = c$; and fuppofing H to be the point required, let $BH = x$, and therefore $HF = c - x$. Now the fquare of $HA = aa + xx$, that of $HC = bb + xx$, that of $HD = bb + cc - 2cx + xx$, and that of $HE = aa + cc - 2cx + xx$. And hence the equation $aa + xx + bb + cc - 2cx + xx = bb + xx + aa + cc - 2cx + xx$. Now as it is an identical equation, the fame as $0 = 0$, which is as much as to fay, that in the right line BF, wherever we take the point H it will always agree to the property required.

Equations and problems how divided.

78. Equations which reduced contain the unknown quantity of one dimenfion only, are called *Simple Equations*, or of the *firft degree*. Thofe which contain the unknown quantity raifed to the fquare, whether they are quadraticks fimple or affected, are faid to be of the *fecond degree*. Thofe which contain the unknown quantity raifed to the cube, however the other terms may be, are faid to be of the *third degree*. And fo accordingly are thofe of the *fourth*, *fifth*, and higher degrees. Moreover, thofe problems which are expreffed by fimple equations, as alfo thofe of the fecond degree, are called *Plane Problems*, becaufe they may be conftructed by the common Geometry of *Euclid*, or by rules and compaffes only. All the others are called *Solid Problems*, becaufe for their conftruction is required the defcription of certain curves, which therefore are called *Solid Places*. I fhall fay nothing here of the Refolution and Conftruction of Solid Problems, intending to treat of them exprefsly in Sect. IV.

Equations may fometimes be depreffed to a lower degree.

79. There are many equations, which at firft fight feem to be of that degree which is intimated by the index of the greateft power of the unknown quantity, which, however, when duly managed may be brought down to an inferiour degree. Of this kind are all thofe in which, befides the firft term, which is that of the higheft power of the unknown quantity, and the term which is entirely known, one other term is contained, in which the unknown quantity afcends to a power which is the fquare-root of the power of the firft term. As if the equation were this, $x^4 - 2aaxx = b^4$; which being managed by the Rule of Affected Quadraticks, is reduced to this, $xx = aa \pm \sqrt{a^4 + b^4}$; and therefore $x = \pm \sqrt{aa \pm \sqrt{a^4 + b^4}}$. After the fame manner, this equation $x^6 + a^3x^3 - b^6 = 0$, being reduced, becomes $x^3 = \frac{-a^3 \pm \sqrt{a^6 + 4b^6}}{2}$, and

therefore

therefore $x = \sqrt[3]{\frac{-a^3 \pm \sqrt{a^6 + 4b^6}}{2}}$; and infinite others of a like nature. There are others of the ſame kind, which by means of the extraction of a root may be brought down to an inferiour degree. Thus $x^4 - 2ax^3 + aaxx - 2bbxx + 2abbx + b^4 = aabb + b^4$, having it's firſt member a ſquare, the root of which is $xx - ax - bb$, may be reduced to a lower equation, $xx - ax - bb = \pm b\sqrt{aa + bb}$. Thus, in the equation $x^3 + 3axx + 3aax = b^3$, if we add a^3 on both ſides, it will be $x^3 + 3axx + 3aax + a^3 = a^3 + b^3$, of which the firſt member is a cube, whoſe root is $x + a$. Therefore the equation reduced lower will be $x + a = \sqrt[3]{a^3 + b^3}$. But it is not always thus eaſy, to know what quantity may be added or ſubtracted to or from the firſt member of the equation, ſo that it may become a perfect power, nor can any method be aſſigned for it; ſo that the induſtry and practice of the analyſt can only be of ſervice in theſe caſes.

80. But, if the propoſed problem ſhould be of ſuch a nature, that one unknown quantity being aſſumed, would hardly or not at all be ſufficient to have all the denominations that are neceſſary for finding the equation; in this caſe may be taken one, two, three, or as many more unknown quantities as are needful. And if the problem be determinate in it's own nature, it will always ſupply conditions for as many equations as are the unknown quantities aſſumed. Then, by means of each of theſe equations, one of the unknown quantities will be eliminated, or it's value may be found by the remaining and the given quantities; ſo that finally we ſhall arrive at the laſt equation, which will contain one unknown quantity only. The manner of performing theſe operations will be beſt underſtood by the examples.

Problems will often require more unknown quantities than one.

Firſt, let there be two ſimple equations, or of the firſt degree; as, ſuppoſe for example $a + x = b + y$, and $2x + y = 3b$; and let us eliminate y, and retain x. Now, by means of which we pleaſe of the two equations, ſuppoſe of the firſt, by the help of proper tranſpoſitions of the terms, we may find the value of y, which will be $y = a + x - b$. This value may be ſubſtituted inſtead of y in the ſecond, and we ſhall have a new equation $2x + a + x - b = 3b$, that is $x = \frac{4b - a}{3}$. And this value being ſubſtituted inſtead of x in either of the two propoſed equations, we ſhall have the value of $y = \frac{2a + b}{3}$. This may alſo be obtained by deriving two values of y from the two equations, and comparing them together. For from the firſt equation we ſhall have $y = a + x - b$; and from the ſecond, $y = 3b - 2x$; wherefore it will be, by compariſon, $a + x - b = 3b - 2x$, and thence $x = \frac{4b - a}{3}$, as before.

81. After

How they are to be eliminated.

81. After the ſame manner we muſt proceed, when the equations contain the unknown quantity, which is to be eliminated, raiſed to the ſecond dimenſion; if by means of one of the two given equations, or by the tranſpoſition of the terms alone, or by the rule for ſimple or affected quadraticks, we can have a value to be ſubſtituted in the other equation. Let the two equations be $xx + 5ax = 3yy$, and $2xy - 3xx = 4aa$. Now if we would eliminate y, the ſecond equation will give $y = \frac{4aa + 3xx}{2x}$, and therefore $yy = \frac{16a^4 + 24aaxx + 9x^4}{4xx}$. This value being ſubſtituted in the firſt equation, it will be $xx + 5ax = \frac{48a^4 + 72aaxx + 27x^4}{4xx}$; which, by reduction, will be $23x^4 - 20ax^3 + 72aaxx + 48a^4 = 0$. But if we would eliminate x, finding it's value by either of the two equations, for example by the ſecond, we ſhould have $x = \frac{y}{3} \pm \frac{\sqrt{yy - 12aa}}{3}$. This being ſubſtituted in the firſt equation, it will become $\frac{2yy - 12aa \pm 2y\sqrt{yy - 12aa}}{9} + \frac{5ay \pm 5a\sqrt{yy - 12aa}}{3} = 3yy$. This being freed from radicals, and ſet in order, after a long calculation will come out $69y^4 - 90ay^3 + 72aayy + 40a^3y + 316a^4 = 0$.

Quantities to be eliminated by compariſon.

82. Often by two equations, in which the unknown quantity to be eliminated is raiſed in both to the ſame degree, may be found by means of either of them the value of the higheſt power of the unknown quantity; and that is by putting that higheſt power alone on one ſide of the equation, and all the other terms on the other ſide: then theſe two values being compared to each other, will give an equation of a lower degree. The ſame operation may be repeated again, and ſo on, till we have an equation truly ſimple in reſpect of the unknown quantities, and conſequently it's value expreſſed by the other unknown quantity, and by ſuch as are known. Then this value being ſubſtituted in one of the given equations inſtead of the unknown quantity and it's powers, we ſhall have an equation expreſſed by the other unknown quantity only, and ſuch as are known.

Let the two equations be $y^3 + aay = bxx$, and $y^3 - bxx = aax$, out of which y is to be eliminated. Therefore by the firſt it will be $y^3 = bxx - aay$, and by the ſecond, $y^3 = aax + bxx$. Then by compariſon, $bxx - aay = aax + bxx$, or $y = -x$. Then making a due ſubſtitution in either of the two equations, we ſhall have $-x^3 - aax = bxx$, or $x^2 + bx = -aa$. Again, let the two equations be $xx + 5ax = 3yy$, and $2xy - 3xx = 4aa$, from which we are to eliminate x. It will be by the firſt $xx = 3yy - 5ax$, and by the ſecond, $xx = \frac{2xy - 4aa}{3}$. Therefore the equation will be $3yy - 5ax = \frac{2xy - 4aa}{3}$.

From

From hence we ſhall have $x = \frac{9yy + 4aa}{2y + 15a}$; and this value being ſubſtituted in one of the propoſed equations, in the firſt for inſtance, it will be as is found above.

But if in the two equations the unknown quantity to be eliminated do not aſcend to the ſame power in the higheſt terms, the equation of the lower degree is to be multiplied by ſuch a power of the ſame quantity, that it may be of the ſame degree as the other; and then you are to proceed as before. Thus, if we have $y^3 = xyy + 3aax$, and $yy = xx - xy - 3aa$, and we are to expunge y; multiply the ſecond equation by y, and it will be $y^3 = xxy - xyy - 3aay$. Therefore $xyy + 3aax = xxy - xyy - 3aay$, which, being compared with the value of yy given by the ſecond propoſed equation $yy = xx - xy - 3aa$, will give $\frac{xxy - 3aay - 3aax}{2x} = xx - xy - 3aa$, or $3xxy - 3aay + 3aax = 2x^3$, and therefore $y = \frac{2x^3 - 3aax}{3xx - 3aa}$; which being ſubſtituted in one of the propoſed equations, ſuppoſe in the ſecond, will be $\frac{4x^6 - 12aax^4 + 9a^4xx}{9x^4 - 18aaxx + 9a^4} = xx - 3aa - \frac{2x^4 - 3aaxx}{3xx - 3aa}$; or reducing to the ſame denominator, $x^6 + 18a^2x^4 - 45a^4x^2 + 27a^6 = 0$.

In particular caſes particular expedients may often be uſed, and there may be more expedite methods of coming to a concluſion; but theſe do not fall under any rule. An example may be ſeen of this in theſe two equations, $x + y + \frac{yy}{x} = 20b$, and $xx + yy + \frac{y^4}{xx} = 140bb$. If we would eliminate x we muſt tranſpoſe y in the firſt equation, which will then be $x + \frac{yy}{x} = 20b - y$; and ſquaring both parts, it will be $xx + 2yy + \frac{y^4}{x^2} = 400bb - 40by + yy$, that is $xx + yy + \frac{y^4}{x^2} = 400bb - 40by$. But the firſt member of this equation is the ſame as that of the ſecond propoſed equation, and therefore it will be $400bb - 40by = 140bb$, or $y = \frac{13b}{2}$.

83. By a calculation more laborious and long, but performed after the ſame manner, if there be three, four, or more equations, and as many unknown quantities, we may reduce them to one only. For by means of one equation we may exterminate one unknown quantity, the value of which, expreſſed by the others and known quantities, may be ſubſtituted in every one of the remaining equations. Then by means of another equation we may eliminate another unknown

When there are ſeveral equations.

unknown quantity, and it's value may be fubftituted in thofe that remain; and fo on to the end. Let there be three equations $x + y = c + z$, $z + x = a + y$, $z + y = b + x$, and we would have only one equation including z. From the firft equation take the value of y, that is $y = c + z - x$, and fubftitute this value in the other two, which are then $z + x = a + c + z - x$, and $z + c + z - x = b + x$, or rather $2x = a + c$, and $2z = b - c + 2x$, which will then be in the place of the fecond and third. In this laft, inftead of $2x$ fubftitute it's value from the other, and then it will be $2z = b - c + a + c$, that is $z = \frac{a + b}{2}$. Alfo, the fame may be done after another manner, thus. From each of the three equations given take the value of y, for example, that is $y = c + z - x$, $y = z + x - a$, $y = b + x - z$. By the comparifon of two and two of thefe equations, which you pleafe, you will form two equations which have no y. From one of which equations you may take the value of one of the unknown quantities, and fubftitute it in the other. Thus, if you make the two equations $c + z - x = z + x - a$, and $c + z - x = b + x - z$, from the firft take the value of x, or $x = \frac{a + c}{2}$, and fubftitute it in the fecond; then $c + z - \frac{a + c}{2} = b + \frac{a + c}{2} - z$; that is, $z = \frac{a + b}{2}$, as above. In the fame manner we muft proceed if the given equations be more in number, and more compounded. The ufe of the rules here taught will be feen in the folution of the problems.

Sometimes the number of equations may be infufficient.

84. Whenever the conditions, or the data of the problem, do not fupply us with as many equations as are the unknown quantities affumed, but that two of them will at laft remain; the problem will always be indeterminate, and we cannot find the value of one of the unknown quantities but on fuppofing and determining the value of the other; in which cafe every indeterminate problem becomes determinate. To give fome idea of thefe indeterminate problems, though by way of anticipation; let it be propofed to feek two numbers, the fum of which is equal to 30. I call the firft number x, the fecond will be $30 - x$ by the condition of the problem, nor fhall I then have any means of forming another equation. Then I will call the fecond y, and by the condition of the problem it will be $x + y = 30$. Now becaufe it is not poffible to find matter for another equation, by which to eliminate one of the two unknown numbers, the problem of it's own nature will be indeterminate. But if I affign a determinate value to one of the unknown quantities, and fuppofe, for example, that $y = 8$, then it will be $x = 30 - y = 22$. But becaufe we may affign infinite values to y fucceffively, the values of x will alfo be infinite, and confequently the problem is capable of an infinite number of folutions. I will take another example of this from Geometry. Let it be propofed to find a rectangle

rectangle equal to a given ſquare. Let y be the baſe of the rectangle required, it's height x, and aa the given ſquare. Then I ſhall have the equation $aa = xy$; and not having matter for another equation, the problem remains indeterminate; there being in fact infinite rectangles equal to the given ſquare, the baſe may be varied infinitely, and the height alſo relatively to it. But if I add this condition to it, that the baſe, for example, ſhall be equal to half the height, or $\frac{1}{2}x$, then it will be $y = \frac{1}{2}x$, and the equation will be $\frac{1}{2}xx = aa$. And thus one of the unknown quantities may be varied an infinite variety of ways, and likewiſe the other, ſo that the problem may have an infinite number of ſolutions.

85. On the contrary, if the conditions of the problem, which are to be fulfilled, ſhall ſupply us with more equations than there are unknown quantities, the problem will be more than determinate, and by that means may become impoſſible. For, in order to be poſſible, the values of the given quantities muſt be reſtrained to a given law, which will often afford innumerable caſes in which the problem will become poſſible. In the foregoing example, of finding two numbers the ſum of which ſhall be 30, when nothing more is required, it will be an indeterminate problem; but if the condition be added, that beſides the difference of the ſquares of thoſe numbers ſhall be given, ſuppoſe for example 60, the problem will then be determinate, we having in this caſe two equations, that is, $x + y = 30$, and $xx - yy = 60$; ſo that, taking from the firſt the value of y, and ſubſtituting it's ſquare in the ſecond, it will be $x = \frac{960}{60}$, or $x = 16$, and conſequently $y = 14$. But beſides, if we ſhould annex a third condition, that the ſum of the ſquares of theſe numbers ſhould be equal to a given number, the problem is more than determinate, and therefore poſſible in one caſe only, in which the number given for the ſum of the ſquares is juſt the ſame as thoſe ſquares, that is 452. Thus, in the other example of a rectangle equal to a given ſquare, if we require that the rectangle ſhould be upon a given baſe, the problem will be determinate; but more than determinate if we ſhould alſo require, that it's ſides ſhould have a given ratio to each other. It will be poſſible only in one caſe, wherein this ratio is exactly the ſame as reſults from the other condition of the given baſe, and from the equality to the given ſquare.

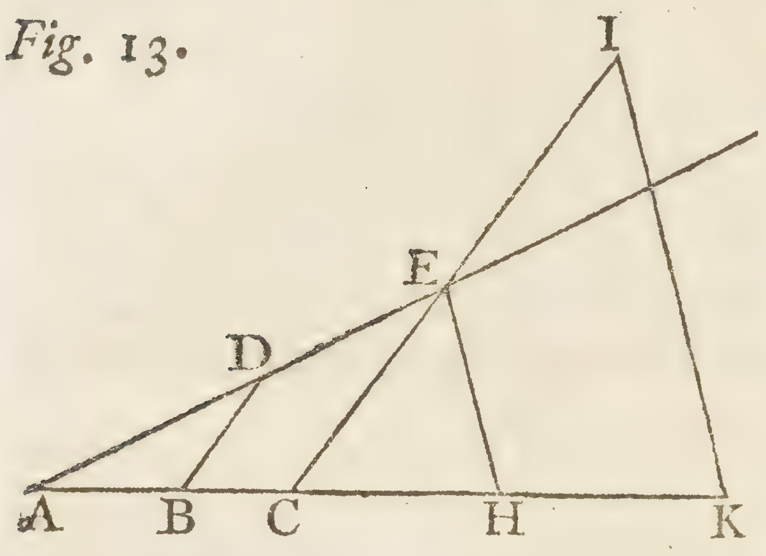
More equations may be given than ſufficient, and the problem become impoſſible.

86. The equations being reſolved, and the values of the unknown quantities being found in geometrical problems, it remains to give the conſtructions of theſe values; that is, from the given lines of the problem we muſt find ſuch, that may exactly repreſent the unknown quantities, which were propoſed to be found. In the firſt place, let the value of the unknown quantity be a ſimple

How ſimple equations may be conſtructed geometrically.

rational fraction, such as $x = \frac{ab}{c}$. If we convert this into an analogy, it will be $c \,.\, b :: a \,.\, x$; so that the fourth proportional required is $\frac{ab}{c}$. Therefore,

Fig. 13.



upon the indefinite right line AC taking AB $= c$, and at any angle drawing BD $= b$, and through the points A, D, drawing the indefinite line AE; if we make AC $= a$, and draw CE parallel to BD, it will be CE $= \frac{ab}{c} = x$. Or else in any angle EAC drawing the indefinite right lines AE, AC, if we take AB $= c$, AD $= b$, AC $= a$, and from the point B to the point D draw the right line BD; from the point C draw CE parallel to BD, it will be AE $= \frac{ab}{c}$. Therefore by these or other theorems or problems of Geometry may be found a fourth proportional to the three given quantities, or a third if only two be given; and we shall have the value of the unknown quantity expressed by lines. If it be $x = \frac{abc}{mn}$, the first analogy is had by taking any one of the letters of the denominator, and two of the numerator; for example, $m \,.\, b :: a \,.\, \frac{ab}{m}$, which is therefore the fourth. Then let this be found as before, and call it f; therefore it will be $x = \frac{fc}{n}$. The second analogy then will be thus, $n \,.\, f :: c \,.\, x = \frac{fc}{n}$, which will be the fourth $= \frac{abc}{mn}$. Taking therefore (Fig. 13.) AB $= m$, AC $= a$, BD $= b$, it will be CE $= \frac{ab}{m} = f$; whence producing CE indefinitely, take CH $= n$, CK $= c$, and draw HE; if from the point K the right line KI be drawn parallel to HE, it will be CH . CE :: CK . CI; that is, $n \,.\, \frac{ab}{m} :: c \,.\, \frac{abc}{mn} =$ CI $= x$.

If the dimensions in the numerator and denominator shall be more in number, the analogies must also be more, but always in the same order.

Or if they consist of several terms.

87. Whence if the value of the unknown quantity shall be compounded of several simple fractions, or of integers and fractions; find the lines which are equal to each term, and adding or subtracting them according to their signs, they will give the line which expresses the value of the unknown quantity.

88. From

88. From this rule we may derive a method of transforming any plane into another with a given ſide; a ſolid into another with one or two given ſides, &c.; that is, any term of two, three, or more dimenſions, into another which ſhall include any given letter, if it be of two dimenſions; or one or two given letters, if it be of three dimenſions. Thus let the term be bb which we deſire to tranſform into another, which ſhall include the letter a. By this letter a let bb be divided, and it will be $\frac{bb}{a}$. By the given rule (Fig. 13.) a line may be found equal to $\frac{bb}{a}$, which call m. Then is $\frac{bb}{a} = m$, and therefore $bb = am$. Let ffc be ſo transformed as that it may include ab. A line may be found equal to $\frac{ffc}{ab}$, which call n. Then it will be $\frac{ffc}{ab} = n$, or $ffc = abn$. If it had been required that it ſhould only include a, we ſhould have made $\frac{fc}{a} = n$, and therefore $\frac{ffc}{a} = fn$, or $ffc = afn$. This is manifeſt, and needs no other examples.

How the terms of an equation may be transformed at pleaſure, and ſo fitted for conſtruction.

89. This being ſuppoſed, let the value of the unknown quantity be a complicate fraction, or more than one, that is, let the denominator have ſeveral terms; as $x = \frac{a^3}{bb + cc}$. One of the terms, ſuppoſe cc, is to be transformed into another, which ſhall include the letter b, and let it be bm. Then we ſhall have $\frac{a^3}{bb + bm}$, which is reſolved into theſe two analogies, $b \,.\, a :: a \,.\, \frac{aa}{b}$, the fourth, and $b + m \,.\, \frac{aa}{b} :: a \,.\, \frac{a^3}{bb + bm}$, the other fourth. And making as uſual the conſtruction by the help of ſimilar triangles, we ſhall have the line which is the value of the unknown quantity x. We might as well have left the term cc in the denominator, and have transformed bb into another, which ſhould have included the letter c, for example cn; then the fraction would have been $\frac{a^3}{cc + cn}$, which is reſolved into theſe analogies, $c \,.\, a :: a \,.\, \frac{aa}{c}$, and $c + n \,.\, \frac{aa}{c} :: a \,.\, \frac{a^3}{cc + cn}$. Let the fraction given be $x = \frac{b^3c}{a^3 + b^3}$; in the denominator the term b^3 may be transformed into aan, and the quantity to be conſtructed will be $\frac{b^3c}{a^3 + a^2n}$. This may be reſolved into three analogies, $a \,.\, b :: b \,.\, \frac{bb}{a}$, and $a \,.\, b :: \frac{bb}{a} \,.\, \frac{b^3}{a^2}$, and $a + n \,.\, c :: \frac{b^3}{a^2} \,.\, \frac{b^3c}{a^3 + a^2n}$. If the denominator ſhould have three terms, then perhaps two of them muſt be transformed; if it ſhould have

How complicate terms may be tranſformed.

have four, then three are to be transformed, &c. Thus, if there were given $x = \frac{b^3c}{a^3 + b^3 - bcc}$, after having made $b^3 = aan$, and $bcc = aap$, then it would be $x = \frac{b^3c}{a^3 + a^2n - a^2p}$. This, in the ſame manner, is reſolved into three analogies, $a . b :: b . \frac{bb}{a}$, $a . b :: \frac{bb}{a} . \frac{b^3}{a^2}$, and $a + n - p . c :: \frac{b^3}{a^2} . \frac{b^3c}{a^3 + a^2n - a^2p} = \frac{b^3c}{a^3 + b^3 - bcc}$.

It can make no difficulty if the numerator of the fraction ſhould be complicate, or have ſeveral terms; becauſe the fraction will be equivalent to ſo many fractions as are the terms of the numerator. Thus $\frac{aa \pm bb}{a^3 - c^3}$ is the ſame as $\frac{aa}{a^3 - c^3} \pm \frac{bb}{a^3 - c^3}$. Therefore each being reſolved in the manner here explained, the ſum or difference of the lines ſo found, according as their ſigns may require, will give the line which is the value of the unknown quantity required.

Other fractions conſtructed.

90. But without multiplying operations, by reducing a fraction with a complicate numerator to ſeveral fractions, it will be enough to make uſe of a convenient transformation of the terms of the numerator and denominator, after the ſame manner as has already been ſeen for the denominator. Thus let it be $x = \frac{aa + bc}{a + b}$; transform the term bc into am, and the fraction will be $\frac{aa + am}{a + b}$; whence it is $a + b . a + m :: a . \frac{aa + am}{a + b}$. Let it be $\frac{aacc - abcf}{acf + bff}$; make $bf = am$, and the fraction will be $\frac{aacc - aacm}{acf + amf}$, that is $\frac{acc - acm}{cf + mf}$; then $f . a :: c . \frac{ac}{f}$, and $c + m . c - m :: \frac{ac}{f} . \frac{acc - acm}{fc + mf}$.

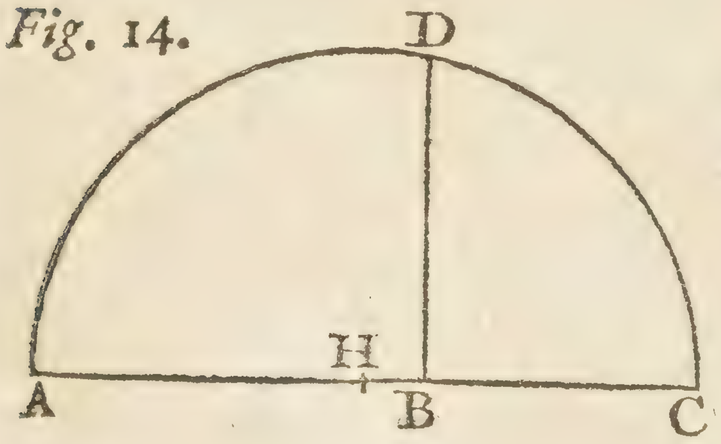
But if the numerator and denominator of the fraction be ſuch, that without transforming any term they may be reſolved into their linear components; then no uſe is to be made of transformation, which would only multiply operations unneceſſarily. Such will be the fractions $\frac{aab}{aa - cc}$, $\frac{a^3 - ab^2}{ac + cc}$; and ſuch others. The firſt of theſe may be reſolved into theſe two analogies, $a + c . a :: a . \frac{aa}{a+c}$, and $a - c . b :: \frac{aa}{a + c} . \frac{aab}{aa - cc}$. And the ſecond into theſe two, $c . a :: a + b . \frac{aa + ab}{c}$, and $a + c . a - b :: \frac{aa + ab}{c} . \frac{a^3 - abb}{ac + cc}$. Thus very often, without

without transforming the terms, it will be more convenient to make uſe of the Extraction of Roots, for reſolving a fraction into analogies. Thus the fraction $\frac{aa+bc}{a}$ may be reſolved into this analogy, $a \,.\, \sqrt{aa+bc} :: \sqrt{aa+bc} \,.\, \frac{aa+bc}{a}$; though more ſimply thus, $a + \frac{bc}{a}$. The fraction $\frac{a^3+abb}{aa+cc}$ is reſolved into theſe two analogies, $\sqrt{aa+cc} \,.\, \sqrt{aa+bb} :: \sqrt{aa+bb} \,.\, \frac{aa+bb}{\sqrt{aa+cc}}$, and $\sqrt{aa+cc} \,.\, a :: \frac{aa+bb}{\sqrt{aa+cc}} \,.\, \frac{a^3+abb}{aa+cc}$. Yet ſometimes it may be neceſſary to transform a term; as in the fraction $\frac{a^3+bbc}{aa-cc}$, which cannot be reſolved even by radicals, unleſs one of the terms of the numerator be transformed, ſuppoſe bbc into acm, ſo that it may be $\frac{a^3+acm}{aa-cc}$. For then it may be $a+c \,.\, a :: \sqrt{aa+cm} \,.\, \frac{a\sqrt{a^2+cm}}{a+c}$; and $a-c \,.\, \sqrt{aa+cm} :: \frac{a\sqrt{aa+cm}}{a+c} \,.\, \frac{a^3+acm}{aa-cc} = \frac{a^3+bbc}{aa-cc}$. The ſame obtains in fractions more compounded.

Among the variety of ways here produced, it cannot eaſily be determined which will be beſt in particular caſes; perhaps more than one ſhould be tried, that we may pitch upon that which will furniſh out the ſimpleſt conſtruction of the propoſed problem.

91. As to what concerns the finding ſuch lines as are expreſſed by radicals; in the third place, let the value of the unknown quantity be an integer quadratick radical, ſuppoſe $x = \sqrt{ab}$. That is, x is a mean proportional between a and b. Take AB $= a$, and directly to it BC $= b$, and biſecting the compoſed line AC in H, with radius HC deſcribe the ſemicircle ADC, and from the point B raiſe the perpendicular BD terminated at the circumference. The rectangle of AB into BC will be equal to the ſquare of BD; that is, $ab =$ BDq, and therefore $\sqrt{ab} =$ BD $= x$. Let it be $x = \sqrt{2aa}$; taking AB $= 2a$, and BC $= a$, it will be BD $= \sqrt{2aa}$, &c. Radicals how conſtructed.

Fig. 14.



And if the radical conſiſted of complex quantities, as $x = \sqrt{4aa \pm ab}$, or elſe $x = \sqrt{3aa \pm ab \pm 2ac}$; in the firſt caſe, making AB $= 4a \pm b$; and in the ſecond, AB $= 3a \pm b \pm 2c$, and taking BC $= a$; if a ſemicircle ADC be deſcribed upon the diameter AC, and a perpendicular BD be raiſed, that perpendicular

pendicular in the firſt caſe will be equal to $\sqrt{4aa \pm ab} = x$, and in the ſecond, $\sqrt{3aa \pm ab \pm 2ac} = x$.

And, in general, let the terms under the *vinculum* be as many as you pleaſe, and combined with their ſigns in any manner, it's value may always be conſtructed by means of a ſemicircle, when every term is multiplied into the ſame letter; making, for example, one of the ſegments CB equal to that letter, and the other ſegment BA equal to the ſum or difference of all the terms divided by that letter, and raiſing the perpendicular BD. It is eaſy to perceive, that if the combination of the ſigns ſhould make the ſegment BA a negative quantity, that then the quantity under the *vinculum* would be negative, and therefore that the value of the unknown quantity would be imaginary. Such would be $x = \sqrt{ab - ac}$, ſuppoſing c to be greater than b.

How radicals are to be transformed, in order to conſtruction.

92. Now if every term be not multiplied by the ſame letter, they may become ſuch by transforming thoſe that are not ſo. Thus, if $x = \sqrt{aa \pm bb}$, make $bb = am$, and it will be $x = \sqrt{aa \pm am}$. Then taking $AB = a \pm m$, that is $AB = a \pm \frac{bb}{a}$, and $BC = a$, and deſcribing the ſemicircle, it will be $BD = \sqrt{aa \pm bb} = x$. In like manner, having given $x = \sqrt{aa + bb - cc}$, make $bb = am$, $cc = an$, and it will be $\sqrt{aa + am - an} = x$; and taking $AB = a + m - n$, and $BC = a$, it will be $BD = \sqrt{aa + bb - cc} = x$.

Quadraticks conſtructed without transformation.

93. But however the terms may be, without making any alteration, quadratick radicals may always be conſtructed, either by a right-angled triangle alone, or by that and a circle together. Let it be $x = \sqrt{aa + bb}$, and take $AB = a$, and $BC = b$ perpendicular to AB, it will be $AC = \sqrt{aa + bb} = x$. If $x = \sqrt{2aa}$, make $AB = a$, and $BC = a$, and it will be $AC = \sqrt{2aa}$. If $x = \sqrt{3aa}$, make, as at firſt, $AB = BC = a$, and upon the right line AC raiſing the perpendicular $CD = a$, it will be $AD = \sqrt{3aa}$. If $x = \sqrt{5aa}$, make $AB = 2a$, $BC = a$, then $AC = \sqrt{5aa}$. If $x = \sqrt{aa + bb + cc}$, make $AB = a$, $BC = b$ and perpendicular to AB, and upon AC raiſe the perpendicular $CD = c$; then the hypothenuſe AD will be $= x = \sqrt{aa + bb + cc}$; and ſo on to quantities more compounded. If $x = \sqrt{aa + bc}$, though the term bc be not transformed,

Fig. 15.

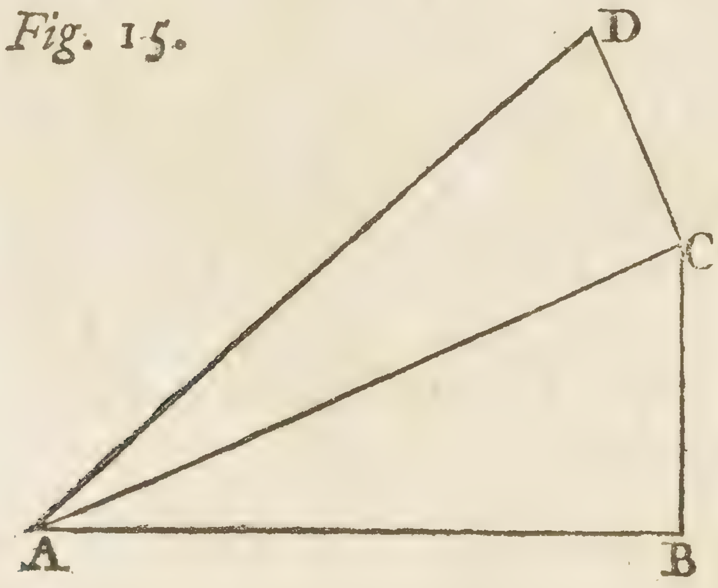
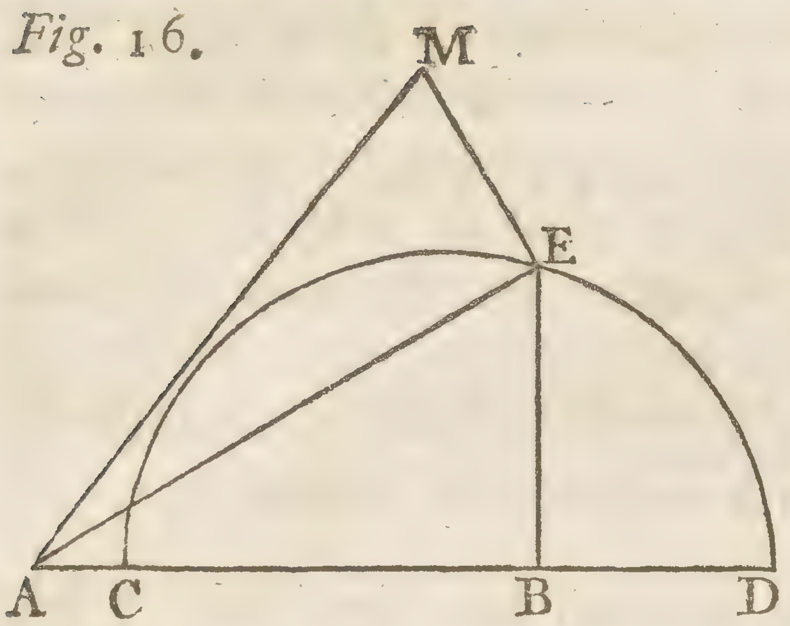


Fig. 16.



transformed, in the manner shown above, taking AB $= a$, BC $= b$, BD $= c$, upon the diameter CD describe the semicircle CED, then the ordinate BE will be $= \sqrt{bc}$; and drawing the hypothenuse AE, it will be $= x = \sqrt{aa + bc}$. If $x = \sqrt{aa + bc + ee}$, upon AE draw the perpendicular EM $= e$, and it will be AM $= x = \sqrt{aa + bc + ee}$. Let $x = \sqrt{aa + bc + cc}$, taking BC $= b + c$, BD $= c$, it will be BE $= \sqrt{bc + cc}$, and AE $= \sqrt{aa + bc + cc}$. If there should be more terms, the operations might increase, but not the difficulty. Let $x = \sqrt{aa - bb}$; on the diameter AB $= a$, let the semicircle ACB be described, in which inscribe the chord AC $= b$; then, because of the right angle ACB, it will be BC $= \sqrt{aa - bb}$. If $x = \sqrt{aa - bb + hh}$, produce AC to M, so that it may be CM $= h$; and drawing BM, it will be $= x = \sqrt{aa - bb + hh}$. If $x = \sqrt{aa - bb - hh}$, in the semicircle ACB inscribe the chord AC $= \sqrt{bb + hh}$; then BC $= \sqrt{aa - bb - hh}$. If $x = \sqrt{aa - bc}$, or $x = \sqrt{aa - bc - ce}$; taking AB $= b$ in the first case, and $= b + e$ in the second, add directly AD $= c$, AH $= a$, if with the diameters BD, AH, be described the two semicircles BCD, AEH; the ordinate AC in the first case will be $= \sqrt{bc}$, and in the second $= \sqrt{bc + ce}$, and therefore, taking AE $=$ AC, and drawing the chord EH, it will be $\sqrt{aa - bc}$ in the first case, and $= \sqrt{aa - bc - ce}$ in the second. If it were $x = \sqrt{aa - bc - ee}$, make AB $= b$, AD $= c$, and besides, taking CF $= e$ perpendicular to AC, it will be AF $= \sqrt{bc + ee}$. Wherefore, making AI $=$ AF, it will be IH $= x = \sqrt{aa - bc - ee}$.

Fig. 17.

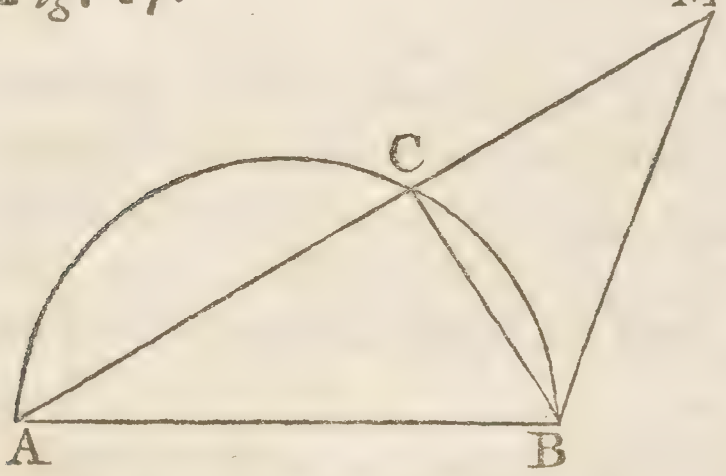
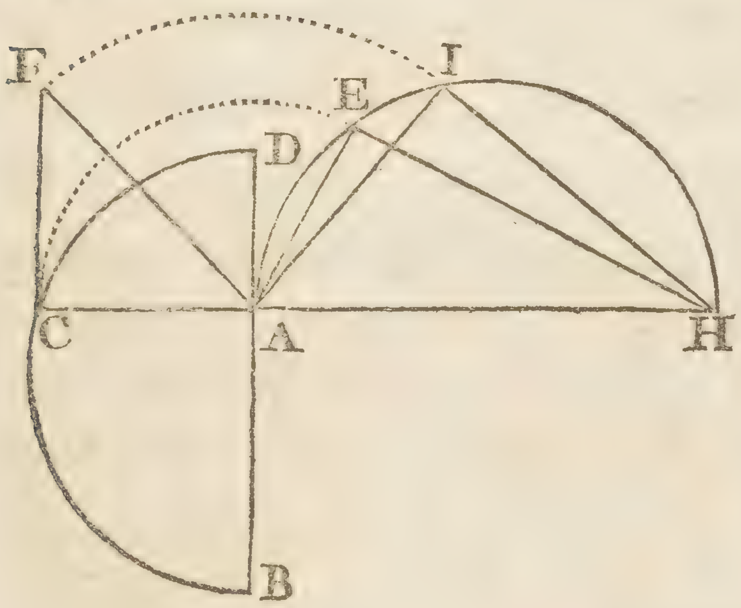


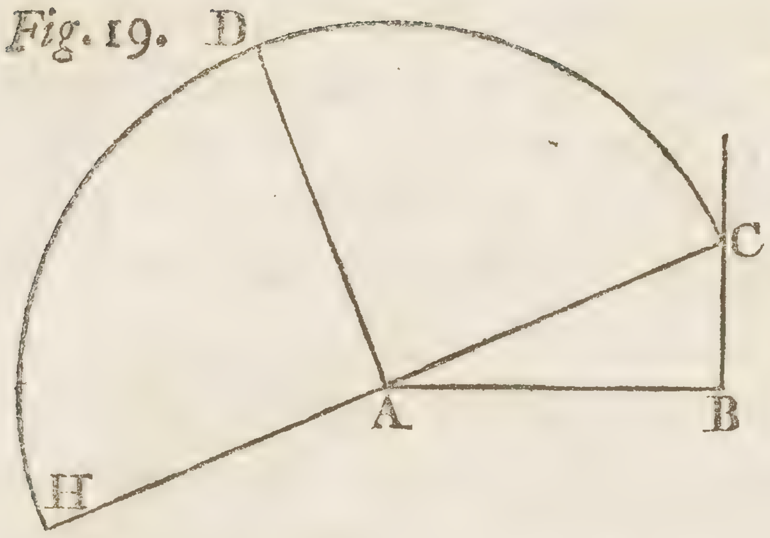
Fig. 18.



If $x = \sqrt[4]{a^4 + b^4}$, that is $x = \sqrt{\sqrt{a^4 + b^4}}$, transform the second term b^4 into $aamm$, and it will be $x = \sqrt{\sqrt{a^4 + a^2m^2}}$; and taking the square aa out of

the

Fig. 19.



the fecond radical, it will be $x = \sqrt{a\sqrt{aa + mm}}$. Make AB $= a$, and the normal BC $= m$, it will be AC $= \sqrt{aa + mm}$. Produce CA to H, fo that it may be AH $=$ AB $= a$. Upon the diameter HC defcribe the femicircle HDC, and from the point A draw AD perpendicular to the diameter; it will be AD $= \sqrt{a\sqrt{aa + mm}} = \sqrt[4]{a^4 + b^4} = x$.

Cafes more compounded are eafily reducible to thefe here fpecified. I fhall add nothing about fractions compounded with rational quantities or radicals, becaufe they require nothing more than applying, or perhaps extending, the rules already given.

Affected quadraticks how conftructed, independent of their folution.

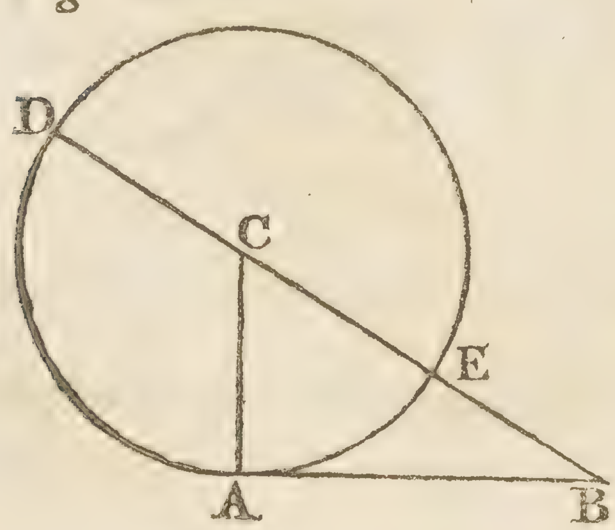
94. As to the conftruction of affected quadratick equations, which are the higheft I intend to treat of in this Section, I thought their refolution to be neceffary, and have given rules by which to obtain the values of the unknown quantity, and fo to conftruct them in the manner juft now taught. Yet this previous refolution is not abfolutely neceffary, and without this they may be conftructed after the following manner.

All the infinite number of affected quadratick equations may be comprehended and expreffed by this formula, $xx \pm ax \pm bb = 0$, that is, by thefe four, which arife from the four different combinations of their figns.

1. $xx + ax - bb = 0$.
2. $xx - ax - bb = 0$.
3. $xx + ax + bb = 0$.
4. $xx - ax + bb = 0$.

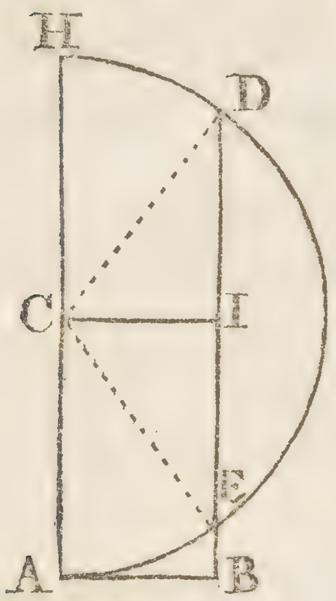
It is to be underftood, that the letter a reprefents the whole quantity which forms the co-efficient of the fecond term; and b is the fquare-root of the aggregate of all the known terms. Now to contract the two firft, take CA $= \frac{1}{2}a$, AB at right angles to it, and equal to b. With radius CA let a circle AED be defcribed, and from the point B let the right line BD be drawn, terminating in the periphery at D, and paffing through the centre C. Then will BE be the pofitive value of the unknown quantity, in the equation $xx + ax - bb = 0$, and BD it's negative value: as on the contrary, in the equation $xx - ax - bb = 0$, BD will be the pofitive value, and BE the negative value. And in effect, by refolving the two equations, they

Fig. 20.



they are $x = -\frac{1}{2}a \pm \sqrt{\frac{1}{4}aa + bb}$, and $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa + bb}$. And by the conſtruction, it being $CA = CD = CE = \frac{1}{2}a$, and $AB = b$, it will be $CB = \sqrt{\frac{1}{4}aa + bb}$, and therefore $BE = \sqrt{\frac{1}{4}aa + bb} - \frac{1}{2}a$, which is the poſitive value of the unknown quantity in the firſt equation; and BD taken negatively, $= -\frac{1}{2}a - \sqrt{\frac{1}{4}aa + bb}$, will be the negative value. Thus BD, taken poſitively, $= \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$, is the poſitive value of the unknown quantity in the ſecond equation; and becauſe of CB greater than CE, EB will be negative, $= \frac{1}{2}a - \sqrt{\frac{1}{4}aa + bb}$, which is the negative value.

Fig. 21.



The third and fourth formulas are thus conſtructed. Taking $CA = \frac{1}{2}a$, and AB at right angles equal to b, as in the foregoing conſtruction; and with radius CA deſcribing a ſemicircle ADH; draw BD parallel to AC. The two right lines BE, BD, will be the two values, or the two negative roots of the equation $xx + ax + bb = 0$; and the two poſitive values in the equation $xx - ax + bb = 0$. Now reſolving the equations, the third will give us $x = -\frac{1}{2}a \pm \sqrt{\frac{1}{4}aa - bb}$, and the fourth $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa - bb}$. Therefore, drawing the right lines CD, CE, and CI perpendicular to BD, it will be $ID = IE = \sqrt{\frac{1}{4}aa - bb}$, and therefore BE negative $= -\frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, the negative value of the unknown quantity in the third equation, becauſe BI is greater than IE. And BD taken negative will be $= -\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$, the other negative value in the ſame third equation. On the contrary, BD will be poſitive, $= \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, and BE poſitive, $= \frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$, both being the poſitive values of the unknown quantity in the fourth equation.

Therefore, to conſtruct any affected quadratick equation, it will ſuffice to aſſume the radius CA equal to half the co-efficient of the ſecond term, and the tangent AB equal to the ſquare-root of the laſt term; and the reſt as in one or the other of the two figures, according as the laſt term ſhall be poſitive or negative. Thus, for example, to conſtruct the equation $xx + ax - bx - aa + cc = 0$, make $AC = \frac{a-b}{2}$, and $AB = \sqrt{aa - cc}$ in the firſt of the two figures, if a be greater than c; and $AB = \sqrt{cc - aa}$, in the ſecond, if a be leſs than c. By this example it may be ſeen how we are to proceed in all other caſes.

A caſe may happen, that, in the conſtruction of Fig. 21, the right line BD ſhall not cut, but touch the circle ADH; or that it may neither cut nor touch it. It will touch it when it is $AC = AB$, that is, $\frac{1}{2}a = b$, and the two values of the unknown quantity of the equation, BE, BD, ſhall be equal, one poſitive

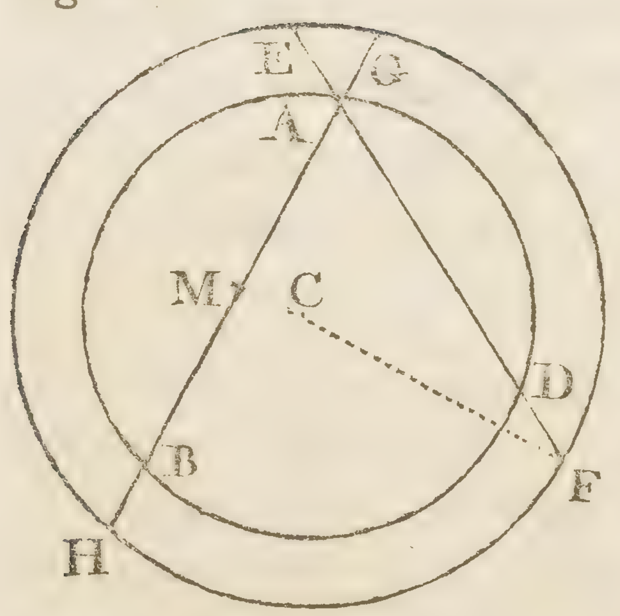
and the other negative. It will neither touch it nor cut it when BA is greater than AC, that is, b greater than $\frac{1}{2}a$; and the unknown quantities will not have any value at all, but will be impoffible or imaginary. And this agrees perfectly with the analytical refolution, becaufe when it is $\frac{1}{2}a = b$, it will be $\frac{1}{4}aa - bb = 0$, and therefore the two values $x = -\frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, and $x = \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, will be $x = -\frac{1}{2}a$, and $x = \frac{1}{2}a$. And when $\frac{1}{2}a$ is lefs than b, then $\sqrt{\frac{1}{4}aa - bb}$ will be an imaginary quantity, and therefore the two values of the unknown quantity will be imaginary.

Or otherwife thus conftructed.

95. In thefe conftructions it is neceffary to find the fquare-root of the laft term of the equation, which is to fupply us with the tangent AB of the circle. If therefore this laft term is equal to a rectangle, or if we have a mind to make it fo, which thing is in our own power, the four formulas aforegoing may be thus conftructed, after another manner.

$$1.\quad xx + ax - bc = 0.$$
$$2.\quad xx - ax - bc = 0.$$
$$3.\quad xx + ax + bc = 0.$$
$$4.\quad xx - ax + bc = 0.$$

Fig. 22.

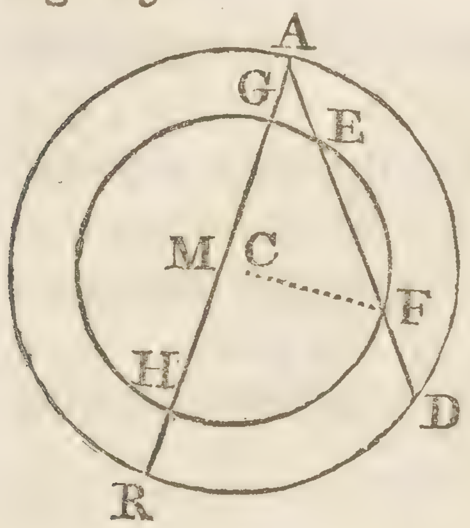


Let the circle BAD be defcribed with any diameter, provided it be not lefs than either a or $b - c$; where I fuppofe b greater than c, or that b is the greater fide of the rectangle given, and c the leffer fide. Now, from any point A in the periphery let the two chords AB $= a$ and AD $= b - c$ be infcribed in the circle, and let this laft be produced to F, fo as that DF $= c$. With centre C of the firft circle, and with radius CF, let a fecond circle FGH be defcribed, which may cut the chords AD, AB produced, in the points F, E, G, H. This being done, AG will be the pofitive value or root, and AH the negative, in the equation $xx + ax - bc = 0$. And on the contrary, AG will be the negative root, and AH the pofitive, in the equation $xx - ax - bc = 0$.

Now, to apprehend the reafon of this, it is neceffary to have recourfe to two properties of the circle, which are demonftrated by geometricians; which are, that the right lines EA, DF, are equal to each other, as alfo the two GA, BH, are equal, and that the rectangles EA $\times$ AF, and GA $\times$ AH are alfo equal. Thefe two theorems being fuppofed, the line BA is to be bifected in M. Then, by *Euclid*, ii. 6, the fquare of MG will be equal to the fquare of MA, together with the rectangle BG $\times$ GA, that is HA $\times$ AG, that is FA $\times$ AE. But the

the ſquare of MA, by the conſtruction, is equal to $\frac{1}{4}aa$, and the rectangle FA $\times$ AE is equal to bc. Therefore it will be MG $= \sqrt{\frac{1}{4}aa + bc}$, and thence AG $= -\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bc}$, the poſitive value. But AH $= \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bc}$, whence AH negative $= -\frac{1}{2}a - \sqrt{\frac{1}{4}aa + bc}$ the other value which is negative; both exactly as they ariſe from the reſolution of the firſt equation. For the ſame reaſon, AG negative will be $= \frac{1}{2}a - \sqrt{\frac{1}{4}aa + bc}$, and AH poſitive $= \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bc}$, which are the values of the unknown quantity in the ſecond equation.

Fig. 23.



As to the third and fourth equation, let any circle RAD be deſcribed with a diameter not leſs than a, or $b + c$. From any point of the periphery A let two chords be inſcribed in it, that is AR $= a$, and AD $= b + c$; and making DF $= c$, with centre C of the firſt circle, and with radius CF, let another circle GHF be deſcribed, which ſhall cut the two chords AR, AD, in the points G, H, F, E. This being done, AG, AH, ſhall be the two negative values in the third equation, and the two poſitive in the fourth. For, biſecting RA in M, it will be, by *Euclid*, ii. 6, the ſquare of MA equal to the rectangle HA $\times$ AG, that is RG $\times$ GA, that is DE $\times$ EA, together with the ſquare of MG. Therefore it will be $\frac{1}{4}aa = bc +$ MGq, or MG $= \sqrt{\frac{1}{4}aa - bc}$. And therefore $-$ MA $+$ MG, that is GA negative, will be $= -\frac{1}{2}a + \sqrt{\frac{1}{4}aa - bc}$. And $-$ MG $-$ MR, that is GR negative, will be $= -\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bc}$, both the negative values of the unknown quantity in the third equation. In like manner, MG $+$ MR, that is $\frac{1}{2}a + \sqrt{\frac{1}{4}aa - bc}$, will be GR poſitive; and MA $-$ MG, that is $\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bc}$, will be AG poſitive, both the poſitive values of the unknown quantity in the fourth equation.

It is plain, both by the conſtruction of Fig. 23, and by the reſolution of the third and fourth equations, that when it is $bc = \frac{1}{4}aa$, the circle HGEF will touch the right line RA, and the two values will be equal. And if bc ſhall be greater than $\frac{1}{4}aa$, it will neither touch it nor cut it, and then the two values will become imaginary.

Having thus laid down the principal rules, I ſhall proceed to ſhow their uſe in the ſolution of ſome particular Problems.

PRO-

PROBLEM I.

An arithmetical problem.

96. Let there be a certain ſum of ſhillings, which is to be diſtributed among ſome poor people; the number of which ſhillings is ſuch, that if 3 were given to each, there would be 8 wanting for that purpoſe; and if 2 were given, there would be an overplus of 3 ſhillings. It is required to know, what was the number of the poor people, and how many ſhillings there were in all.

Let us ſuppoſe the number of poor to be x; then, becauſe the number of ſhillings was ſuch, that, giving to each 3, there would be 8 wanting; the number of ſhillings was therefore $3x - 8$. But, giving them 2 ſhillings a-piece, there would be an overplus of 3; therefore again the number of ſhillings was $2x + 3$. Now, making theſe two values equal, we ſhall have the equation $3x - 8 = 2x + 3$, and therefore $x = 11$ was the number of the poor. And becauſe $3x - 8$, or $2x + 3$, was the number of the ſhillings, if we ſubſtitute 11 inſtead of x, the number of ſhillings will be 25.

PROBLEM II.

A problem of equable motion.

97. The velocities of two bodies being given, their diſtance, and the difference of time in which they begin to move in a right line; the point in that line, and the time is required, in which the bodies will meet.

Fig. 24.

Let the firſt body be at A, the velocity of which is ſuch, that it would deſcribe the ſpace c in the time f. Let B be the ſecond body, with ſuch a velocity, that it would deſcribe the ſpace d in the time g. Let the difference of time in which they begin to move be h, and let their diſtance AB be e. Firſt, let them move the ſame way, and let them come together at the point D. Make $AD = x$, then $BD = x - e$. To obtain an equation it muſt be conſidered, that, having given the difference of time from the beginning of the motion of the body A, and of the body B, the time muſt be found employed by the body A, and alſo by the body B, and to the leſſer of theſe times, or to that of the body which moves laſt, muſt be added the given difference, and then theſe two portions of time ought to be made equal. Therefore, by the rule of proportion, we muſt ſay, if the body A deſcribe the ſpace c in the time f, in what time will it

defcribe the fpace x? That is, $c \,.\, f :: x \,.\, \frac{fx}{c}$, which is therefore the fourth term. Likewife, if the body B defcribe the fpace d in the time g, in what time will it defcribe the fpace $x - e$? That is, $d \,.\, g :: x - e \,.\, \frac{gx - ge}{d}$, which is the fourth term. Therefore the time of the body A is $\frac{fx}{c}$, and the time of the body B is $\frac{gx - ge}{d}$, and their difference is h. And if the body A began to move after the body B, it will be $\frac{fx}{c} + h = \frac{gx - ge}{d}$; and reducing to a common denominator, it will be $fdx + cdh = cgx - cge$, that is, $cgx - fdx = cdh + ceg$; and, dividing by $cg - fd$, it is $\frac{cdh + ceg}{cg - df} = x$.

If the body A move before the body B, it will be $\frac{fx}{c} = h + \frac{gx - ge}{d}$; and reducing to a common denominator, it is $dfx = cdh + cgx - ceg$, that is, $cgx - dfx = ceg - cdh$. And, dividing by $cg - df$, it is $x = \frac{ceg - cdh}{cg - fd}$. Now, if inftead of x we fubftitute it's value now found, in the expreffion of the whole time $\frac{fx}{c} + h$ in the firft cafe, and in $\frac{fx}{c}$ in the fecond, we fhall have the time required.

I fhall apply the formula to fome examples. Let the body A have fuch a velocity, as to move 9 miles in 1 hour, and the body B to move 15 miles in 2 hours; and let them be diftant from each other 18 miles, and let B begin to move 1 hour before A. Then it will be $h = 1$, $f = 1$, $c = 9$, $g = 2$, $d = 15$, $e = 18$; and therefore $x = \frac{324 + 135}{18 - 15} = 153$. Subftitute this value inftead of x, and alfo the others, in the expreffion of the time $\frac{fx}{c} + h$, and it will be $= 18$. Therefore the two moving bodies will be together at the diftance from the point A of 153 miles, after 18 hours from the beginning of the motion.

Let the body A have fuch a velocity as to move 4 miles in 1 hour, and the body B to move 5 miles in 1 hour, and let them be diftant 6 miles, and A begin to move 2 hours before B. Therefore it will be $h = 2$, $f = 1$, $c = 4$, $g = 1$, $d = 5$, $e = 6$. Taking the formula of the fecond cafe, it will be $x = \frac{24 - 40}{4 - 5} = 16$. And fubftituting this value of x with the others in the

expreffion

expreffion of the time $\frac{fx}{c}$, it will be $= 4$. Therefore the two bodies A and B will be together at the diftance of 16 miles from the point A, after 4 hours from the beginning of the motion.

But if the two bodies move contrary ways, or towards each other, let them meet, for example, in the point M; then calling AM $= x$, and retaining the fame denominations as above, BM only will be changed, which will now be $= e - x$; and confequently the time of the body B to defcribe the fpace BM will be $\frac{ge - gx}{d}$. Wherefore, if A begin it's motion after the body B, it will be $\frac{fx}{c} + h = \frac{ge - gx}{d}$; and if it begin it's motion firft, it will be $\frac{fx}{c} = h + \frac{ge - gx}{d}$; of which equations the firft is $fdx + cdh = cge - cgx$, that is $x = \frac{cge - cdh}{cg + fd}$; and the fecond is $fdx = cdh + cge - cgx$, or $x = \frac{cge + cdh}{fd + cg}$.

Let the body A have fuch a velocity, as to defcribe 7 miles in two hours, and the body B 8 miles in 3 hours, and let them be diftant 59 miles, and A begin to move 1 hour before B. Therefore it will be $h = 1$, $f = 2$, $c = 7$, $g = 3$, $d = 8$, $e = 59$; and therefore, taking the fecond formula $x = \frac{cge + cdh}{cg + fd}$, and fubftituting thefe values, it will be $x = \frac{1239 + 56}{21 + 16}$, that is $x = 35$. Therefore the two bodies will meet each other at the diftance of 35 miles from the point A, after 10 hours from the beginning of motion; as will be feen by fubftituting thefe values in the expreffion $\frac{fx}{c}$, which is the whole time of motion.

PROBLEM III.

Ευρηκα, a famous problem of *Archimedes.*

98. Having given the mafs of the crown of King *Hiero*, made up of a mixture of gold and filver, and the fpecifick gravity of gold, of filver, and of the crown; it is required to find the quantity of each metal in the crown.

Let the mafs of the crown be reprefented by m, the fpecifick gravity of gold to filver be as 19 to $10\frac{1}{3}$, and to the fpecifick gravity of the crown as 19 to 17. Make x the quantity of gold in the crown, and therefore $m - x$ will be the quantity of the filver. The mafs of a body divided by it's denfity or fpecifick gravity

gravity is equal to it's volume; therefore the volume of the crown will be $\frac{m}{17}$, that of the gold $\frac{x}{19}$, and that of the silver $\frac{m-x}{10\frac{1}{3}}$. But the volume of the crown is equal to both the volumes of the gold and silver together which compose it. Therefore we shall have the equation $\frac{m}{17} = \frac{x}{19} + \frac{m-x}{10\frac{1}{3}}$, that is, by reducing it to order, $\frac{19 - 10\frac{1}{3}}{19 \times 10\frac{1}{3}}x = \frac{17 - 10\frac{1}{3}}{17 \times 10\frac{1}{3}}m$, and therefore $x = \frac{6\frac{2}{3} \times 19}{8\frac{2}{3} \times 17}m$, or $x = \frac{190}{221}m$. Hence, supposing, for example, the mass of the crown to be 5 pounds, the quantity of the gold in it will be $4\frac{66}{221}$ pounds, and of the silver $\frac{155}{221}$ parts of a pound.

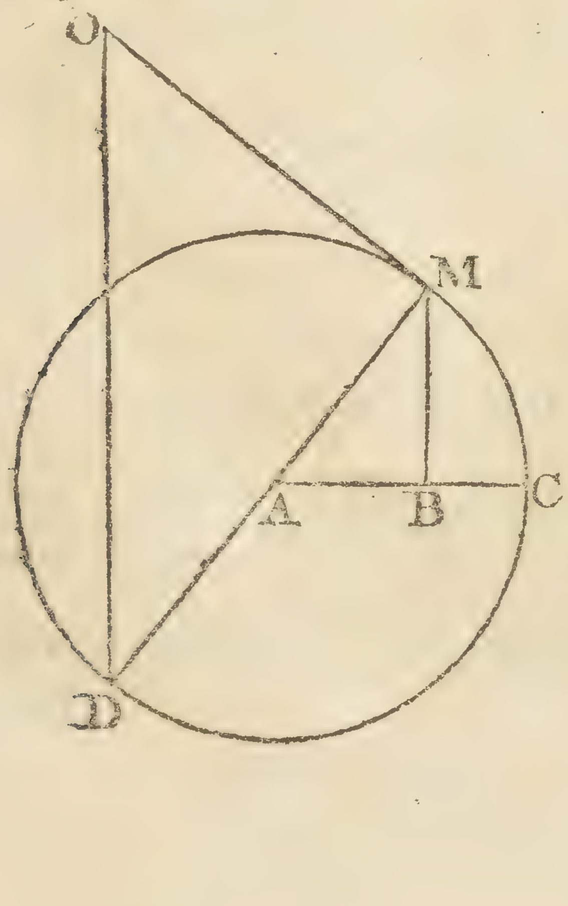
PROBLEM IV.

99. Let there be two weights so related, that if we take from the first 1 pound, the remainder shall be equal to the second weight increased by 1 pound. And, adding 1 pound to the first, and taking 1 pound from the second, the sum shall be double to the remainder. The quantity of each weight is required. An arithmetical problem.

Let us call the first weight x, and the second y. Then it will be $x - 1 = y + 1$ by the first condition, and $\frac{x+1}{2} = y - 1$ by the second. By the first we obtain this value $y = x - 2$, which, substituted in the second, will give $\frac{x+1}{2} = x - 3$, and therefore $x + 1 = 2x - 6$; that is, $x = 7$, and consequently $y = 5$.

PROBLEM V.

Fig. 25.



100. In a given circle DCM, a line AB being given, which is intercepted between the centre and the line MB, drawn from the extremity of the diameter DM perpendicular to AC: it is required to find a point O in the tangent MO, from whence the rectangle of OM into MB may be equal to the rectangle of DM into AB. A geometrical problem.

Make AB $= b$, AM $= a$, MO $= x$; it will be MB $= \sqrt{aa - bb}$, and by the condition of the problem, $x\sqrt{aa - bb} = 2ab$, that is, $x = \frac{2ab}{\sqrt{aa - bb}}$.

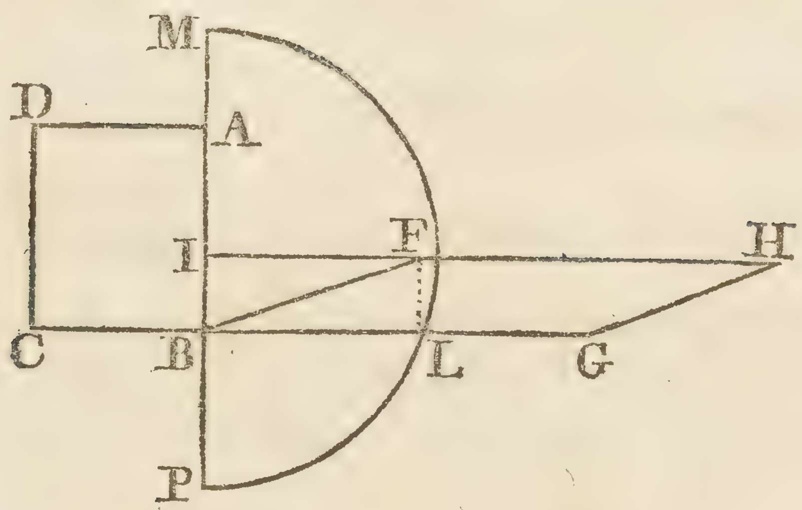
 From

From the point D let there be drawn DO parallel to BM; then the triangles MBA, DMO, will be fimilar, and therefore it will be MB . BA :: DM . MO, that is $\sqrt{aa-bb} . b :: 2a . MO = \frac{2ab}{\sqrt{aa-bb}} = x$.

PROBLEM VI.

Another.

Fig. 26.



101. A rectangle being given, a parallelogram is required, the fides of which are multiples in a given ratio of the fides of the rectangle, and it's area fubmultiple.

Let ABCD be the given rectangle, AB $= a$, BC $= b$, and therefore the area $= ab$. Let the parallelogram required be BFHG, whofe fide BF fhould be to AB as n to e; and therefore BF $= \frac{an}{e}$. The fide BG fhould be to BC as m to e; and therefore GB $= \frac{bm}{e}$. Laftly, the area BFHG fhould be to the given rectangle ab, as e to r. Make BL $= x$, and therefore, drawing FL perpendicular to BG, it will be FL $= \sqrt{\frac{aann}{ee} - xx}$. Wherefore the parallelogram BFHG, that is FL $\times$ BG, will be $\frac{bm}{e}\sqrt{\frac{aann}{ee} - xx}$. And, fince this fhould be to the rectangle ABCD as e to r, we fhall have the analogy $\frac{bm}{e}\sqrt{\frac{aann}{ee} - xx} . ab :: e . r$; whence the equation $\frac{bmr}{ee}\sqrt{\frac{aann}{ee} - xx} = ab$. And taking away the radical, it will be $\frac{aann}{ee} - xx = \frac{aae^4}{mmrr}$, that is $xx = \frac{aann}{ee} - \frac{a^2e^4}{m^2r^2}$; and extracting the fquare-root, $x = \pm\sqrt{\frac{aann}{ee} - \frac{aae^4}{mmrr}}$.

In the fide BA take BI $= \frac{aee}{mr}$, and IM $= \frac{an}{e}$; and with centre I, radius IM, defcribe the femicircle MLP. The ordinate will be BL $= \sqrt{\frac{aann}{ee} - \frac{aae^4}{mmrr}} = x$. Then from the point L raifing the perpendicular LF $=$ BI, and drawing

 BF,

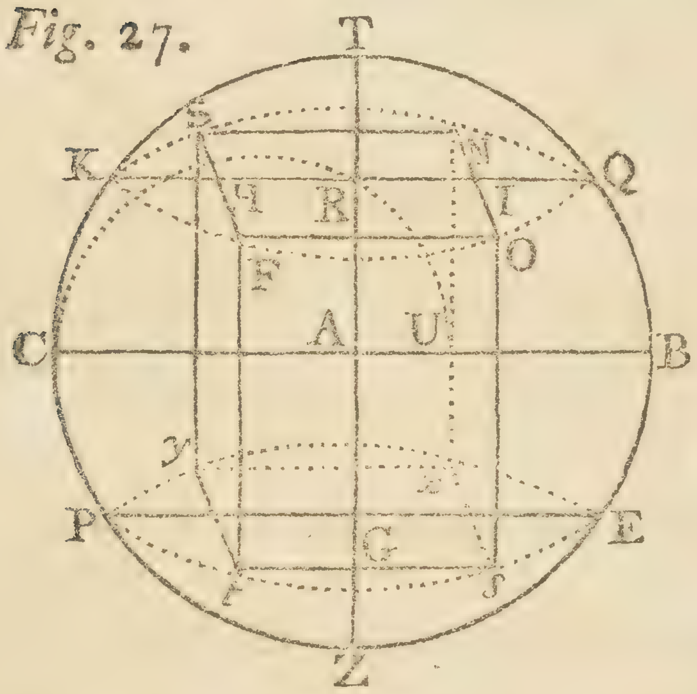
BF, take BG $= \frac{bm}{e}$, and compleating the parallelogram BHFG, it will be $=$ BG $\times$ FL $= \frac{abe}{r}$; that is, it will be to the rectangle BADC $= ab$, as e to r. And the ſide BF will be equal to $\sqrt{BLq + LFq} = \frac{an}{e}$: which was to be conſtructed.

The extraction of the ſquare-root has introduced an ambiguity of ſigns, and therefore two values of the unknown quantity, and conſequently two ſolutions of the problem. But it is eaſy to perceive, that theſe two values are the ſame, and differ from each other only in this, that the ſame conſtruction may be made on the ſide of B towards C.

PROBLEM VII.

102. To inſcribe a cube in a given ſphere.

Fig. 27.



Let KQEP be a great circle of the ſphere, A it's centre, AT $= a$ it's radius, AR half of the height, or of the ſide of the cube to be inſcribed, and therefore make AR $= x$. Through the point R let there be conceived to paſs a plane perpendicular to AT, the common ſection of which, with the ſphere, ſhall be the circle QNSKFO, and the ſquare inſcribed in this circle ſhall be one face, or one plane of the parallelopiped inſcribed in the ſphere. But, becauſe this parallelopiped ought to be a cube, it will therefore follow, that GR $=$ SN $=$ NO, or AR $=$ RI $=$ IO; and beſides, that the planes which incloſe it ſhould be at right angles. In the circle KPEQ the ordinate will be KR $=$ QR $= \sqrt{aa - xx}$; and taking RI $=$ RA $= x$, it will be KI $= \sqrt{aa - xx} + x$, and IQ $= \sqrt{aa - xx} - x$. And in the circle NKOQ, the ordinate IO $= \sqrt{KI \times IQ} = \sqrt{aa - 2xx}$. Therefore the equation will be $\sqrt{aa - 2xx} = x$, and thence $aa = 3xx$, or $x = \pm\sqrt{\frac{1}{3}aa}$. Now, taking AU equal to a third part of the radius AB, upon the diameter CU deſcribe the ſemicircle CRU; the point R in which it cuts the radius AT ſhall be the point required. And it will be AR $= \sqrt{\frac{1}{3}aa}$, half the ſide of the cube, taking it's poſitive value on the ſide of T, and the negative towards Z. Whence taking AG $=$ AR, and through the points R, G, the ſphere being cut by two planes

A geometrical problem.

perpendicular

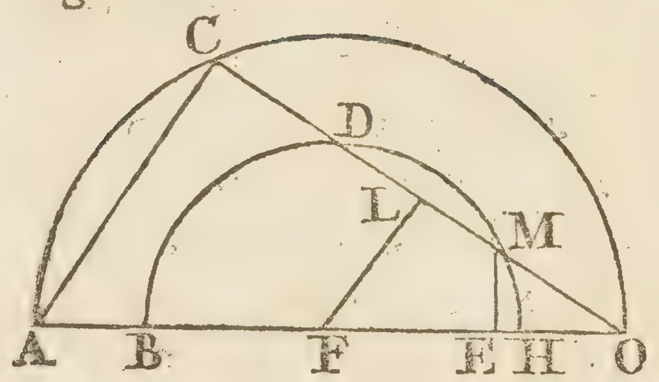
perpendicular to RG; and taking $RH = RI = RA$, and through the points I, H, the ſphere being cut by two other planes perpendicular to HI, and by two others through SN, FO, perpendicular to NO, the cube will be inſcribed. For, becauſe, by the conſtruction, as it plainly appears, the planes are perpendicular to one another, and it being $AR = RI = \sqrt{\frac{1}{3}aa}$, it will be, by the property of the circle KQEP, the ordinate $RQ = \sqrt{\frac{2}{3}aa}$, and therefore $IQ = \sqrt{\frac{2}{3}aa} - \sqrt{\frac{1}{3}aa}$, and $IO = \sqrt{KI \times IQ} = \sqrt{\frac{1}{3}aa}$; and conſequently all the ſides are equal, as was to be demonſtrated.

From the conſtruction of this problem ariſes a pretty ſimple ſynthetical demonſtration. Since AU is a third part of the radius AC, the rectangle CAU, that is the ſquare of AR, will be a third part of the ſquare of the radius, and therefore $AR = RI$. If from the centre A of the ſphere be drawn a right line AI to the point I, the ſquare of AI will be double the ſquare of AR, that is, two third parts of the ſquare of the radius. And if from the ſaid centre A a radius AO be ſuppoſed to be drawn, the ſquare of IO will be equal to the ſquare of AO, leſſened by the ſquare of AI; that is, equal to the ſquare of the radius, leſſened by two third parts of the ſame ſquare, and therefore equal to one third part of the ſquare of the radius, and conſequently IO is equal to AR, &c.

PROBLEM VIII.

Another, producing an identical equation.

Fig. 28.



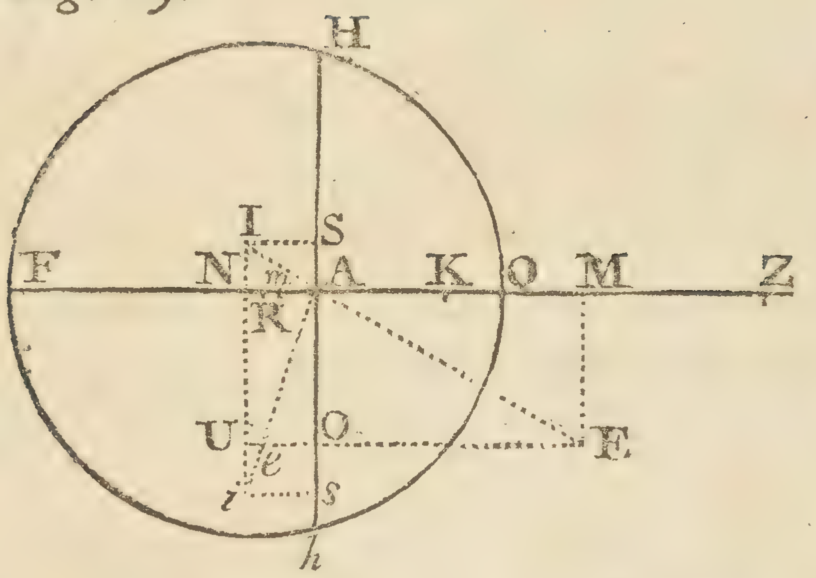
103. Two concentric circles ACO, BDH, being given, from the point O to draw a chord in ſuch manner, that it may be $OM = DC$.

Let OC be the chord required, and let F be the centre. Make $FH = a$, $FO = b$, and letting fall the perpendicular ME to AO, let $FE = x$. Then $EM = \sqrt{aa - xx}$, $EO = b - x$, and therefore $OM = \sqrt{aa - 2bx + bb}$. From the point C draw CA to the extremity of the radius FA. Then the two triangles OEM, OCA, will be ſimilar, and therefore $OM . OE :: OA . OC$. That is, $\sqrt{aa - 2bx + bb} . b - x :: 2b . OC = \frac{2bb - 2bx}{\sqrt{aa - 2bx + bb}}$. But, by *Euclid*, iii. 36, it is $DO \times OM = BO \times OH$; and therefore $DO . BO :: OH . OM$; that is $DO = \frac{\overline{a+b} \times \overline{b-a}}{\sqrt{aa - 2bx + bb}}$. And conſequently $CD = CO - DO = \frac{bb - 2bx + aa}{\sqrt{aa - 2bx + bb}} = \sqrt{bb - 2bx + aa}$. But, by the condition of the problem, it ought to be $OM = CD$. Therefore

it will be $\sqrt{bb - 2bx + aa} = \sqrt{aa - 2bx + bb}$, which is an identical equation. Whence we gather, that, however we may draw the chord OC from the point O, it will always be OM = CD. And this may alſo be known, by drawing from the centre F the perpendicular FL to any chord whatever OC. For F being the centre of both the circles, the right line FL will biſect both DM and CO; and therefore, if from the equals LC, LO, we take the equals LD, LM, there will remain equals CD, MO.

PROBLEM IX.

Fig. 29.



104. The indefinite right line NZ being propoſed, and three points N, A, K, being given in it, a fourth point M is required, ſuch that NM may be a third proportional to NK, AM. A geometrical, or rather arithmetical, problem.

Becauſe the three points N, A, K, are given, make NA $= a$, NK $= b$, AM $= x$, and therefore MN $= a + x$. Then, by the condition of the problem, we ſhall have $b \,.\, x :: x \,.\, a + x$; and, reducing this analogy to an equation, it will be $xx = ab + bx$, or $xx - bx = ab$, which is an affected quadratick. Wherefore, if we add to each ſide the ſquare of half the co-efficient of the ſecond term, that is $\frac{1}{4}bb$, it will be $xx - bx + \frac{1}{4}bb = ab + \frac{1}{4}bb$; and extracting the ſquare-root, it is $x - \frac{1}{2}b = \pm\sqrt{ab + \frac{1}{4}bb}$, that is $x = \frac{b \pm \sqrt{4ab + bb}}{2}$.

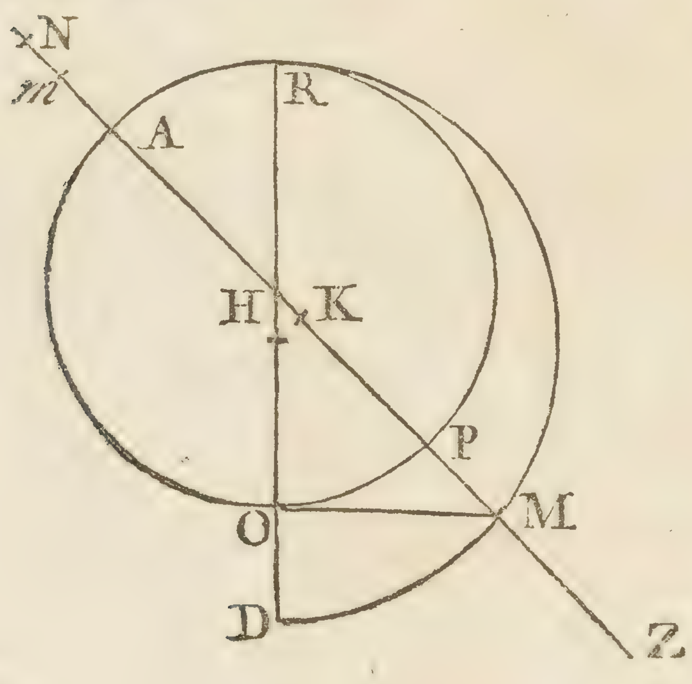
On the right line NZ produced both ways indefinitely, take AR, AQ, equal to each other, and each equal to NK $= b$; and RF four times NA, or RF $= 4a$. Then it will be AF $= 4a + b$. With the diameter FQ let a ſemicircle FHQ be deſcribed; at the point A the ordinate will be AH $= \sqrt{4ab + bb}$. Then adding directly AO = NK $= b$, and biſecting OH in S, it will be OS $= \frac{b + \sqrt{4ab + bb}}{2} = x$. Then taking AM = OS, the point required will be M, as to the poſitive root. For, deſcribing the rectangles SN, AU, MO, and drawing the diagonals AI, AE; becauſe it is OS $= \frac{b + \sqrt{4ab + bb}}{2}$, it will be AS $= \frac{\sqrt{4ab + bb} - b}{2}$, and the rectangle OS × SA will be equal to ab, that is, equal to the rectangle OA × AN. Therefore the ſides of theſe rectangles will be

be to one another in a reciprocal ratio, that is, OA . OS :: SA . AN, or EM . MA :: IN . NA. Wherefore the two lines IA, AE, will be directly to each other, and consequently the triangles IUE, AOE, will be similar, and therefore it will be AO . OE :: IU . UE; but AO = NK, OE = AM, IU = OS = AM, UE = NM. Wherefore NK . AM :: AM . NM.

The foregoing construction belongs only to the positive value of the unknown quantity, that radical being taken which is affected by the affirmative sign. But, in a like manner, that will be constructed in which the sign is negative. For the other semicircle F*h*Q being described, and drawing the ordinate A*h*, it will be $Oh = b - \sqrt{4ab + bb}$, a negative quantity; and bisecting O*h* in S, it will be $Os = \frac{b - \sqrt{4ab + bb}}{2} = x$. So that x is a negative quantity, and therefore, taking $Am = Os$ from A towards F, m will be the other point which solves the problem. For, because it is $As = Ah - sh = \frac{-b - \sqrt{4ab + bb}}{2}$, it is therefore $Os \times sA = ab = OA \times AN$; so that, making the rectangle N*s*, and drawing the diagonal A*i*, because $As \times sO = OA \times AN$, and $AN = si$, it will be A*s* . *si* :: AO . O*e*, and therefore O*s* = O*e*. But O*s* = A*m*, therefore U*e* = N*m*. But, by the similar triangles AO*e*, *i*U*e*, we shall have AO . O*e* :: *i*U . U*e*, and it is AO = NK, *i*U = O*s* = O*e* = A*m*. Therefore it will be NK . A*m* :: A*m* . *m*N.

Without resolving the equation $xx - bx - ab = 0$, the problem may be constructed independently, by the help of § 94, in the following manner. Take RO = NK = b, and directly to it OD = NA = a. Then with the diameter RD let the semicircle RMD be described; the ordinate will be $OM = \sqrt{ab}$. With the diameter OR let another circle ARPO be described, and from the point M through the centre H let the right line MN be drawn. And taking AN = a, NK = b, AM will be the positive value of the unknown quantity. And taking the part A*m* = P*m* from A towards N, A*m* will be the negative value. I omit the construction of the same equation by means of § 95, because it is evident enough of itself.

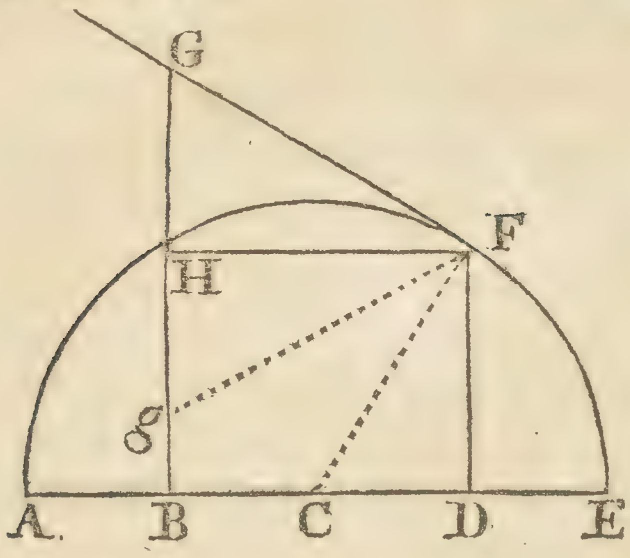
Fig. 30.



PRO-

PROBLEM X.

Fig. 31.



105. The diameter AE of the circle AFE being given, and the two portions CB, CD, from the centre C, and raising the perpendiculars DF, BH; in BH produced, such a point G is required, that, drawing the right line GF to the point F, the rectangle GF $\times$ FD may be equal to the rectangle AC $\times$ BD. A geometrical problem.

Draw FH parallel to AE, and make the radius CA $= r$, CB $= a$, CD $= b$; it will be DF $= \sqrt{rr - bb} =$ BH, and make HG $= x$. Therefore HF $=$ CB $+$ CD $= a + b$, and GF $= \sqrt{aa + 2ab + bb + xx}$. Then, by the condition of the problem, we shall have $\sqrt{aa + 2ab + bb + xx} \times \sqrt{rr - bb} = ar + br$, and, to take away the asymmetry, it will be $a^2r^2 + 2abr^2 + b^2r^2 = a^2r^2 + 2abr^2 + b^2r^2 + r^2x^2 - a^2b^2 - 2ab^3 - b^4 - b^2x^2$, and, by reducing, $r^2x^2 - b^2x^2 - a^2b^2 - 2ab^3 - b^4 = 0$. That is, $x^2 = \frac{a^2b^2 + 2ab^3 + b^4}{r^2 - b^2}$; and, extracting the square-root, it is $x = \pm \frac{ab + bb}{\sqrt{rr - bb}}$. Therefore x, the quantity sought, is a fourth proportional to FD, DC, and FH. Now, because the angles in D and H are right, if we make the angles GFH, gFH, each equal to the angle CFD, the triangles GFH, gFH, CFD, will be similar, and the points G, g, (that is G in respect to the positive value, and g in respect of the negative value,) will satisfy the question. For it will be FG (or Fg) . FH :: FC . FD. But FH $=$ BD, FC $=$ AC; so that it will be GF (gF) . BD :: AC . FD. And therefore GF (gF) $\times$ FD $=$ BD $\times$ AC.

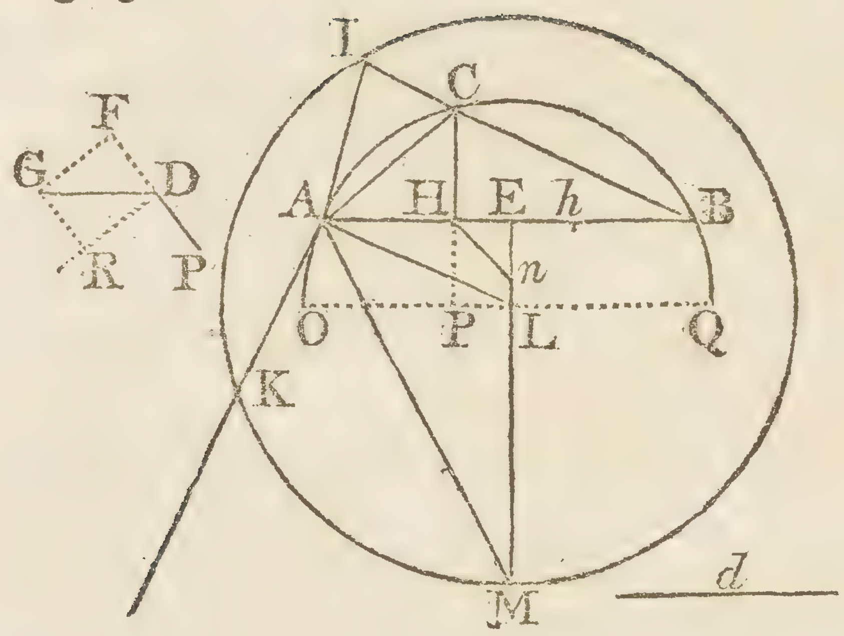
It is easy to perceive, that, in respect of the positive value, it is enough to draw the tangent FG at the point F, because the angles GFC, HFD, are right angles. And taking away the common HFC, the angles GFH and CFD will be equal.

PRO-

PROBLEM XI.

A geometrical problem.

Fig. 32.



106. From the extremities of the given line AB, to draw two right lines AC, BC, in ſuch a manner, that they may make the angle ACB equal to the given angle GDP; and that the ſum of the ſquares of AC and BC may be to the triangle ABC, in the given ratio of $4d$ to a.

Let AB be biſected in E, and letting fall the perpendicular CH, make EH $= x$, HC $= y$. Now, becauſe the problem is determinate, and here are taken two unknown quantities, it will be neceſſary to find two equations. Make EA $= a$, then it will be AH $= a - x$, HB $= a + x$; therefore the ſquare of AC will be $aa - 2ax + xx + yy$, and the ſquare of CB will be $aa + 2ax + xx + yy$, and the triangle ACB $= ay$; but, by the ſecond condition of the problem, the ſum of theſe ſquares ſhould be to the triangle ABC in the given ratio of $4d$ to a; therefore we ſhall have $2aa + 2xx + 2yy \,.\, ay :: 4d \,.\, a$, and thence the equation $aa + xx + yy = 2dy$. Beſides, the angle ACB ought to be equal to the given angle GDP, and therefore, PD being produced, if the angle GDP be obtuſe, and taking GD at pleaſure, draw GF perpendicular to PF; then the angle GDF will be known, the angle GDP being given. And, becauſe alſo DG is known, which was taken at pleaſure, the two lines will be given, DF which make $= b$, and GF, which make $= c$. Then, from the point A draw AI perpendicular to BC produced, the two triangles GDF, ACI, will be ſimilar. Now, becauſe of the ſimilar triangles BCH, BAI, we ſhall have AI $= \frac{2ay}{\sqrt{aa + 2ax + xx + yy}}$, BI $= \frac{2aa + 2ax}{\sqrt{aa + 2ax + xx + yy}}$, and therefore CI $= \frac{aa - xx - yy}{\sqrt{aa + 2ax + xx + yy}}$. And now, becauſe it muſt be CI . AI :: DF . FG, we ſhall have $\frac{aa - xx - yy}{\sqrt{aa + 2ax + xx + yy}} \,.\, \frac{2ay}{\sqrt{aa + 2ax + xx + yy}} :: b \,.\, c$; and thence the ſecond equation $2aby = aac - cxx - cyy$.

To eliminate one of the two unknown quantities; from the two equations (by § 82.) may be deduced the value of xx, that is, from the firſt $xx = 2dy$

 $- yy$

$-yy-aa$, and from the ſecond, $xx=aa-yy-\frac{2aby}{c}$. Whence the equation $2dy-yy-aa=aa-yy-\frac{2aby}{c}$. That is, $dy=aa-\frac{aby}{c}$, or (making $\frac{ab}{c}=f$) $y=\frac{aa}{d+f}$, which is a value of y expreſſed by known quantities only. This ſubſtituted inſtead of y in the equation $xx=2dy-yy-aa$, we ſhall have at laſt $xx=\frac{2aad}{d+f}-\frac{a^4}{\overline{d+f}|^2}-aa$, or $xx=\frac{a^2d^2-a^2f^2-a^4}{\overline{d+f}|^2}$, and thence $x=\pm\frac{a\sqrt{dd-ff-aa}}{d+f}$, a value expreſſed by given quantities only.

Draw AK indefinitely, making the angle KAB equal to the given angle GDP; and from the point E let fall the indefinite perpendicular EM, and from the point A the right line AL perpendicular to AK. Then making DR perpendicular to PD, the angle RDG will be equal to the angle DGF. In like manner, the angle LAE will be equal to the ſame DGF, and beſides, the angles at E and F are right ones. Therefore the triangles LAE, GDF, will be ſimilar, and thence EL $=\frac{ab}{c}=f$, and AL $=\sqrt{aa+ff}$. In EL produced take LM $=d$, and with centre L, radius LM, let a circle be deſcribed, which ſhall cut AK in K. And, becauſe the angle KAL is a right one, the ordinate will be AK $=\sqrt{dd-ff-aa}$. Whence, making En = AK, and drawing MA, and nH parallel to it from the point n, it will be ME . EA :: nE . EH; that is, $d+f\,.\,a::\sqrt{dd-ff-aa}\,.\,\frac{a\sqrt{dd-ff-aa}}{d+f}$ = EH $=x$. This being done, with centre L, and radius LA, let a circle OCQ be deſcribed, and at the point H raiſing the perpendicular CH, draw CA, CB, and ACB ſhall be the triangle required. For, by *Euclid*, iii. 32, the angle ACB is equal to the angle KAE, that is, by the conſtruction, to the angle GDP; and, by the property of the circle, PC $=\sqrt{\text{OP}\times\text{PQ}}=\frac{df+ff+aa}{d+f}$; and therefore HC $=\frac{aa}{d+f}$. And, by making the calculation, we ſhall find, that the ſum of the ſquares of AC and CB is to the triangle ACB preciſely in the ratio of $4d$ to a.

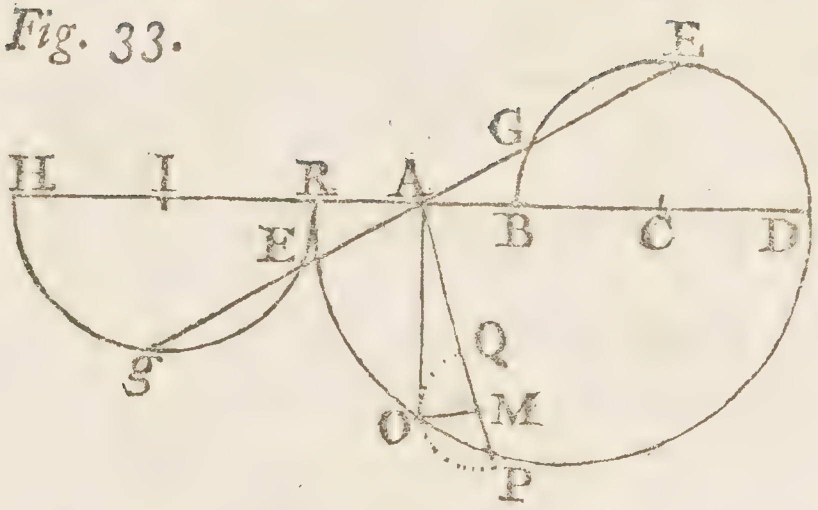
The ambiguous ſign of the final equation gives us two equal values of x, one poſitive, and the other negative. If, therefore, EH taken towards A be conſidered as poſitive, then Eh taken towards B, and equal to EH, will be the negative value; which will require the ſame conſtruction.

It is evident, that the problem will be impoſſible as often as dd is leſs than $ff+aa$, that is, LM leſs than LA; for then the radical will become impoſſible, or only imaginary.

PROBLEM XII.

Another.

Fig. 33.



107. The femicircle BED being given, and a point A being given in the diameter produced; from that point to draw a fecant AE in fuch manner, that the intercepted part GE may be equal to the radius CB.

Make CB $= c$, AB $= b$, AD $= a$, and AG $= x$. Therefore, by the condition of the problem, it will be AE $= c + x$. Now, by *Euclid*, iii. 36, the rectangle EAG is equal to the rectangle DAB, and therefore we fhall have the analogy AE . AD :: AB . AG. That is, $c + x \, . \, a :: b \, . \, x$. Whence the equation $xx + cx = ab$; which is an affected quadratick, and, being refolved as ufual, will give us $x = \pm \sqrt{\frac{1}{4}cc + ab} - \frac{1}{2}c$.

On the right line DA produced, taking AR $=$ AB $= b$, let the femicircle ROD be defcribed on the diameter RD; and drawing the ordinate AO, it will be $= \sqrt{ab}$. Draw OM $= \frac{1}{2}c$ perpendicular to AO, and it will be AM $=$ $\sqrt{\frac{1}{4}cc + ab}$. Then with centre M, and radius MO, let a femicircle QOP be defcribed, and it will be AQ $= \sqrt{\frac{1}{4}c^2 + ab} - \frac{1}{2}c$, the pofitive value of x; and AP $= \sqrt{\frac{1}{4}cc + ab} + \frac{1}{2}c$. Wherefore AP, taken negatively, will be the negative value. Then, if with centre A, and radius AQ, an arch were defcribed, it would cut the femicircle BED in G the point required. And if, on the other fide, the femicircle RGH be defcribed on the diameter RH $=$ BD, an arch on the fame centre, defcribed with radius AP, will cut it in the point required g, which belongs to the negative value. For it being EA $\times$ AG $=$ DA $\times$ AB, that is EA $\times \sqrt{\frac{1}{4}cc + ab} - \frac{1}{2}c = ab$, it will be EA $= \dfrac{ab}{\sqrt{\frac{1}{4}cc + ab} - \frac{1}{2}c}$. And therefore EG $= \dfrac{ab}{\sqrt{\frac{1}{4}cc + ab} - \frac{1}{2}c} - \sqrt{\frac{1}{4}cc + ab} + \frac{1}{2}c$; that is, reducing to a common denominator, EG $= \dfrac{-\frac{1}{2}cc + c\sqrt{\frac{1}{4}cc + ab}}{\sqrt{\frac{1}{4}cc + ab} - \frac{1}{2}c}$. And actually making the divifion, it will be at laft EG $= c$, as it ought to be.

The fame calculus will ferve for the conftruction of the negative value, only making ufe of the rectangle HAR inftead of DAB.

Alfo,

Alſo, the ſolution of the problem may thus be demonſtrated ſynthetically.

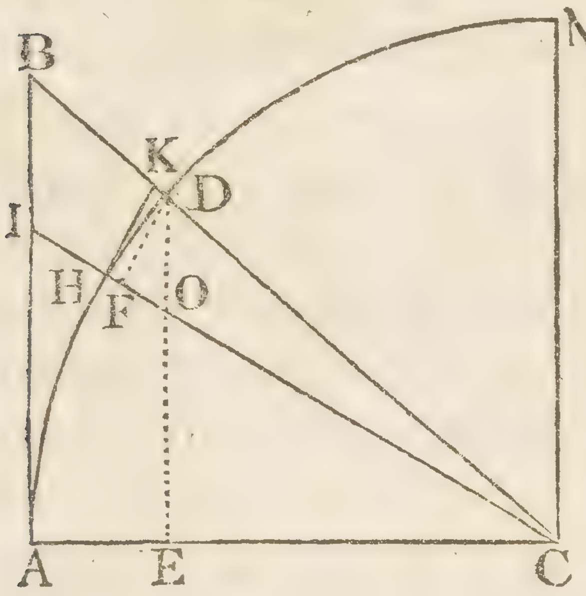
Becauſe it is $OAq = RAD$, and $EAG = DAB$, and, by conſtruction, $AR = AB$, $AQ = AG$, $QP = BC$, $MO = MQ$, it will be $AOq + OMq = AMq = EAG + QMq$; that is, by *Euclid*, ii. 4, $AQq + 2AQM + QMq = EAG + QMq$. And, taking away the common QMq, it will be $AQq + 2AQM = EAG$; and, by the third of the ſame book, $AQq + 2AQM = EGA + GAq$. But $AG = AQ$; therefore it will be $2AQM = EGA$, that is, $AQ . AG :: EG . 2QM$. And therefore $EG = 2QM = BC$.

Q. E. D.

PROBLEM XIII.

108. Two arches of a circle being given, and their tangents, to find the tangent of the ſum of thoſe arches. A trigonometrical problem, with a general ſolution.

Fig. 8.



Let the two given arches be AH, HD, and the tangents $AI = a$, $HK = b$, the radius $CA = r$, the tangent of the ſum of the given arches $AB = x$. It will be $CB = \sqrt{rr + xx}$, $CI = \sqrt{rr + aa}$, $CK = \sqrt{rr + bb}$. And, letting fall DE perpendicular to CA, and DF perpendicular to CH; becauſe of ſimilar triangles CBA, CDE, it will be $CE = \frac{rr}{\sqrt{rr+xx}}$, $DE = \frac{rx}{\sqrt{rr+xx}}$; and alſo, becauſe the triangles CAI, CEO, DFO, are ſimilar, we ſhall have $EO = \frac{ar}{\sqrt{rr+xx}}$, $CO = \frac{r\sqrt{rr+aa}}{\sqrt{rr+xx}}$, and $DO = \frac{b\sqrt{rr+aa}}{\sqrt{rr+bb}}$. Wherefore we ſhall have the equation $ED = EO + OD$, that is, $\frac{ar}{\sqrt{rr+xx}} + \frac{b\sqrt{rr+aa}}{\sqrt{rr+bb}} = \frac{rx}{\sqrt{rr+xx}}$, or $\frac{rx - ar}{\sqrt{rr+xx}} = \frac{b\sqrt{rr+aa}}{\sqrt{rr+bb}}$; and, ſquaring this to free it from the radicals, it will be $\frac{rrxx - 2arrx + aarr}{rr + xx} = \frac{bbrr + aabb}{rr + bb}$. Then, reducing to a common denominator, and taking away ſuch terms as deſtroy one another, it will be $r^4xx - 2ar^4x - 2abbrrx + aar^4 = aabbxx + bbr^4$; that is, $xx - \frac{2ar^4 + 2abbrr}{r^4 - aabb}x = \frac{bbr^4 - aar^4}{r^4 - aabb}$, which is an affected quadratick. Therefore, adding to each member the ſquare of half the co-efficient of the

ſecond

fecond term, that is the fquare of $\frac{ar^4 + ab^2r^2}{r^4 - a^2b^2}$, it will become $xx - \frac{2ar^4 + 2ab^2r^2}{r^4 - aabb}x + \frac{aar^8 + 2aabbr^6 + aab^4r^4}{\overline{r^4 - aabb}|^2} = \frac{b^2r^4 - a^2r^4}{r^4 - a^2b^2} + \frac{a^2r^8 + 2a^2b^2r^6 + a^2b^4r^4}{\overline{r^4 - a^2b^2}|^2}$; then extracting the root, and reducing the *homogeneum comparationis* to a common denominator, it will be $x - \frac{ar^4 + ab^2r^2}{r^4 - a^2b^2} = \pm\sqrt{\frac{b^2r^8 + 2a^2b^2r^6 + a^4b^2r^4}{\overline{r^4 - a^2b^2}|^2}}$. But the quantity under the *vinculum* is a fquare, and it's root is $\frac{br^4 + aabrr}{r^4 - a^2b^2}$, or otherwife $-\frac{br^4 + aabrr}{r^4 - a^2b^2}$. Therefore, in the firft place, taking the pofitive root, it will be $x = \frac{ar^4 + abbrr + aabrr + br^4}{r^4 - aabb}$; and, taking the negative root, it will be $x = \frac{ar^4 + ab^2r^2 - aabr^2 - br^4}{r^4 - aabb}$. Now, in the firft cafe, both the numerator and the denominator are divifible by $rr + ab$, and the quotient is $\frac{arr + brr}{rr - ab}$; and, in the fecond cafe, the numerator and the denominator are divifible by $rr - ab$, and the quotient is $\frac{arr - brr}{rr + ab}$. Therefore the two values of the unknown quantity are $x = \frac{rr \times \overline{a + b}}{rr - ab}$, and $x = \frac{rr \times \overline{a - b}}{rr + ab}$. The firft of thefe will ferve for the tangent of the fum of the given arches, and the fecond for the tangent of their difference, as will eafily be feen by folving the problem in this cafe. This value will be pofitive or negative, according as the arch, or it's tangent a, will be greater or lefs than the tangent b.

This foundation being laid, it will not be difficult to go on to the general folution of the problem; that is, as many fucceffive arches as you pleafe, with their tangents being given, to find the tangent of the fum of all thofe arches; which may be done in the following manner.

Firft, let there be three arches given, and let their tangents be a, b, c. By the foregoing folution, $\frac{rr \times \overline{a + b}}{rr - ab}$ will be the tangent of the fum of two of thofe arches, the tangents of which are a, b. Let this tangent be called z, and therefore it will be $z = \frac{rr \times \overline{a + b}}{rr - ab}$. But, by the fame folution, it will be $\frac{rr \times \overline{z + c}}{rr - zc}$, the tangent of the fum of the two arches, whofe tangents are z, c; and z is the tangent of the fum of the two arches, whofe tangents are a, b.

 Therefore

Therefore $\frac{rr \times \overline{z + c}}{rr - zc}$ will be the tangent of the ſum of the three arches, whoſe tangents are a, b, c. And in this expreſſion, inſtead of z ſubſtituting it's value $\frac{rr \times \overline{a + b}}{rr - ab}$, we ſhall have the tangent of the ſum of the three arches expreſſed by the given tangents only a, b, c, which will be $\frac{rr \times \overline{a + b + c} - abc}{rr - ab - ac - bc}$. By the ſame way of arguing, we ſhall have the tangent of the ſum of four arches, their given tangents being a, b, c, f, which will be

$$\frac{rr \text{ into } \overline{arr + brr + crr + frr - abc - abf - acf - bcf}}{rr \times \overline{rr - ab - ac - af - bc + bf - cf} + abcf}.$$

Alſo, the tangent of the ſum of five, their given tangents being a, b, c, f, g, will be found to be

$$\frac{r^4 \times \overline{a + b + c + f + g} - r^2 \times \overline{abc + abf + acf + abg + bcf + acg + bcg + bff + afg + cfg} + abcfg}{rr \times \overline{rr - ab - ac - af - ag - bc - bf - bg - cf - cg - fg} + abcf + abcg + abfg + acfg + bcfg}.$$

And thus for as many more arches as you pleaſe. From hence may be derived a general rule, to form the fraction which ſhall expreſs the tangent of the ſum of as many given arches as you pleaſe; which will be this.

To form the numerator of the fraction there muſt be taken the ſum of all the poſſible products of an odd number of factors, which can be made with all the given tangents. For example, if the number of tangents be ſeven, take the ſum of all theſe tangents; then the ſum of all the threes that can be made, then the ſum of all the fives, and laſtly, the product of all the ſeven. Theſe ſums are to be multiplied by ſuch a power of the radius, as each has occaſion for, that they may be of a dimenſion greater, by unity, than the number of the given tangents. And to theſe ſums muſt be prefixed the ſigns + and — alternately; that is, to the ſum of all, the ſign +; to the ſum of all the threes, the ſign —, and ſo on; and thus the numerator will be completed.

To form the denominator muſt be taken the ſquare of the radius, then the ſum of all the products of an even number of factors, which can be made by the given tangents, that is of all the twos, of all the fours, &c. This ſquare of the radius, and the ſum of all the twos, of all the fours, of all the ſixes, &c. muſt be multiplied into ſuch a power of the radius, as each has occaſion for, that they may be of a dimenſion equal to the number of the given tangents. To the ſquare of the radius is to be prefixed the ſign +, to all the twos the ſign —, to the fours the ſign +, and ſo on alternately. And thus the denominator will be completed.

Now the rule for knowing what muſt be the number of all the twos poſſible, of all the threes, &c. in a given number of quantities, will be this following.

Write

Write down the number of quantities given, and thence continue the decreafing feries of natural numbers. Under thefe numbers write down in order an increafing feries of natural numbers, beginning from unity. Afterwards find the product of fo many terms of the upper feries, as is the index of the combination that is to be made. Alfo, there muft be made the product of as many terms of the feries below; and one product being divided by the other, the quotient will be the number required. So to know how many twos, threes, &c. can be made of 5 quantities, for example, write down the numbers thus:

$$\begin{array}{ccccc} 5, & 4, & 3, & 2, & 1, \\ 1, & 2, & 3, & 4, & 5. \end{array}$$

The product of the two firft numbers of the upper feries is 20, which, divided by the product of the two firft numbers of the lower feries, will give 10 for the quotient. And therefore the twos will be 10. The product of the three firft is 60, which, divided by 6, the product of the three firft of the lower feries, will give the quotient 10; and therefore the threes will be 10, &c.

From the folution of this problem we obtain, by way of corollary, the folution of another which is more fimple; and that is, the tangent of an arch being given, to find the tangent of any multiple of that arch. For, in this cafe, it will be fufficient to make all the given tangents equal to one another, and equal to the tangent of the given arch. For example, make the tangent of the given arch $= a$, and let it be required to find the tangent of the double arch, the triple, &c. In the formula which we have already found for the tangent of the fum of two given arches, inftead of the letter b we muft every where put a, and we fhall have a formula or expreffion for the double arch $\frac{2arr}{rr - aa}$. In the formula for the tangent of the fum of three given arches, inftead of b and c we muft put a, and we fhall have the expreffion of the triple arch $\frac{3arr - a^3}{rr - 3aa}$. In like manner, that for the quadruple arch will be $\frac{4ar^4 - 4a^3rr}{r^4 - 6a^2r^2 + a^4}$. That for the quintuple arch will be $\frac{5ar^4 - 10a^3r^2 + a^5}{r^4 - 10a^2r^2 + 5a^4}$. And fo of all others fucceffively.

Whence we may form the following progreffion, or general canon, for the tangent of any multiple arch, according to any whole number whatever denoted by n.

$$\frac{nr^{n-1}a - \frac{n.\overline{n-1}.\overline{n-2}}{1.\ 2.\ 3}r^{n-3}a^3 + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.\ 2.\ 3.\ 4.\ 5.}r^{n-5}a^5 - \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.\overline{n-5}.\overline{n-6}}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7}r^{n-7}a^7}{r^{n-1} - \frac{n.\overline{n-1}}{1.\ 2.}r^{n-3}a^2 + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.\ 2.\ 3.\ 4.}r^{n-5}a^4 - \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.\overline{n-5}}{1.\ 2.\ 3.\ 4.\ 5.\ 6}r^{n-7}a^6} \text{ \&c.}$$

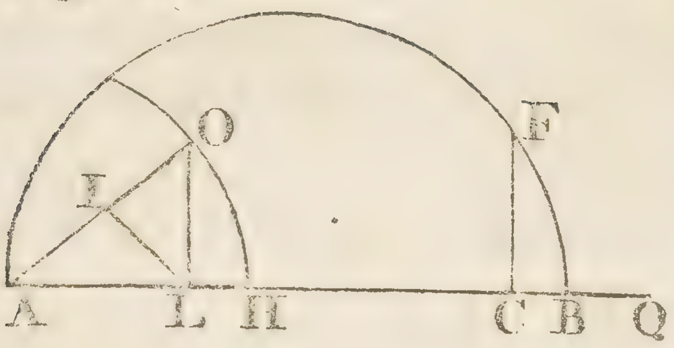
The

The tangent being found of any multiple arch, the inverſe problem will be eaſily reſolved. That is, the tangent of an arch being given, to find the tangent of any ſubmultiple arch, according to any whole number whatever. That is to ſay, to divide an arch or angle into as many equal parts as we pleaſe. Wherefore let the tangent of the given arch be b, and n the number according to which we would have the ſubmultiple arch; we muſt take the tangent found for the multiple arch by the number n, inſtead of a we muſt put x, and thus x will repreſent the tangent of the ſubmultiple arch. This tangent of the multiple arch is therefore equal to the given tangent b, whence we ſhall have an equation to determine the unknown quantity x.

Therefore the tangent b being given, and the radius r, the equation for the tangent of the ſubtriple arch will be $x^3 - 3bxx - 3rrx + brr = 0$. That for the ſubquintuple arch will be $x^5 - 5bx^4 - 10rrx^3 + 10brrxx + 5r^4x - br^4 = 0$. And ſo of the reſt.

PROBLEM XIV.

Fig. 34.



109. To find a triangle ALO, the ſides of which AO, LO, AL, and the perpendicular LI, are in continued geometrical proportion. A geometrical problem.

Take one ſide at pleaſure, or AL $= a$, and make OL $= x$. It will be, by the conditions of of the problem, AO $= \frac{xx}{a}$, and LI $= \frac{aa}{x}$. Therefore AI $= \sqrt{aa - \frac{a^4}{xx}}$, and IO $= \sqrt{xx - \frac{a^4}{xx}}$. Therefore AI + IO = AO, that is, $\sqrt{aa - \frac{a^4}{xx}} + \sqrt{xx - \frac{a^4}{xx}} = \frac{xx}{a}$. Or $\frac{xx}{a} - \sqrt{xx - \frac{a^4}{xx}} = \sqrt{a^2 - \frac{a^4}{xx}}$; and, by ſquaring, $\frac{x^4}{aa} - \frac{2xx}{a}\sqrt{xx - \frac{a^4}{xx}} + xx - \frac{a^4}{xx} = aa - \frac{a^4}{xx}$, that is $\frac{x^4}{aa} + xx - aa = \frac{2xx}{a}\sqrt{xx - \frac{a^4}{xx}}$. Now, by ſquaring again, it will be $\frac{x^8}{a^4} + \frac{2x^6}{aa} + x^4 - 2x^4 - 2aaxx + a^4 = \frac{4x^6}{aa} - 4aaxx$. And laſtly, by reducing to a common denominator, and ordering the equation, it will be $x^8 - 2a^2x^6 - a^4x^4 + 2a^6x^2 + a^8 = 0$. This equation has the appearance of one of the eighth degree, but it may be obſerved to be a ſquare, and therefore, extracting it's root, it will be found to be $x^4 - aaxx - a^4 = 0$. This is an affected

affected quadratick; therefore, transposing $-a^4$, and adding $\frac{1}{4}a^4$ to both sides, and extracting the root by the common rule for affected quadraticks, it will be $xx - \frac{1}{2}aa = \pm \frac{1}{2}\sqrt{5a^4}$, that is, $xx = \frac{1}{2}aa \pm \frac{1}{2}\sqrt{5a^4}$, and finally, $x = \pm\sqrt{\frac{aa \pm \sqrt{5a^4}}{2}}$.

Therefore the unknown quantity will have four values; but it may be observed, that the quantity $\sqrt{5a^4}$ is greater than aa, and therefore, if we take the radical $\sqrt{5a^4}$ negative, that is $-\sqrt{5a^4}$, then the quantity under the common radical *vinculum* will be negative; whence the value of x will be imaginary, and therefore two values will be imaginary, that is $x = \pm\sqrt{\frac{aa - \sqrt{5a^4}}{2}}$. And two will be real, that is $x = \pm\sqrt{\frac{aa + \sqrt{5a^4}}{2}}$, both equal, but one positive and the other negative.

On the indefinite line AQ take AL $= a$, LC $= a\sqrt{5}$, and CB $= \frac{1}{2}a$. Then on the diameter AB describe the semicircle AFB, and erect the perpendicular CF. By the property of the circle, it will be CF $= \sqrt{\frac{aa + aa\sqrt{5}}{2}} = x$. Bisect AC in H, and with centre A, radius AH $= \frac{xx}{a} = \frac{1 + \sqrt{5}}{2}a$, describe the arch HO. From the point L draw LO = CF, and terminated at the arch HO. And if AO be drawn, and the perpendicular LI, then ALO will be the triangle required. For, because it is AL $= a$, LO $= x = \sqrt{\frac{aa + aa\sqrt{5}}{2}}$, AO $=$ AH $= \frac{xx}{a} = \frac{1 + \sqrt{5}}{2}a$; it will be AO . LO :: LO . LA. But the two squares of AL and LO taken together, that is $aa + \frac{aa + aa\sqrt{5}}{2}$, are equal to the square of AO, that is $\frac{6aa + 2a\sqrt{5aa}}{4}$. Wherefore the angle ALO is a right angle, and thence it will be AO . LO :: AL . LI. But, because it is also AO . LO :: LO . LA, it will be likewise LO . LA :: LA . LI. The other negative value, which is equal to the positive, would serve for the construction that may be made under the line AB.

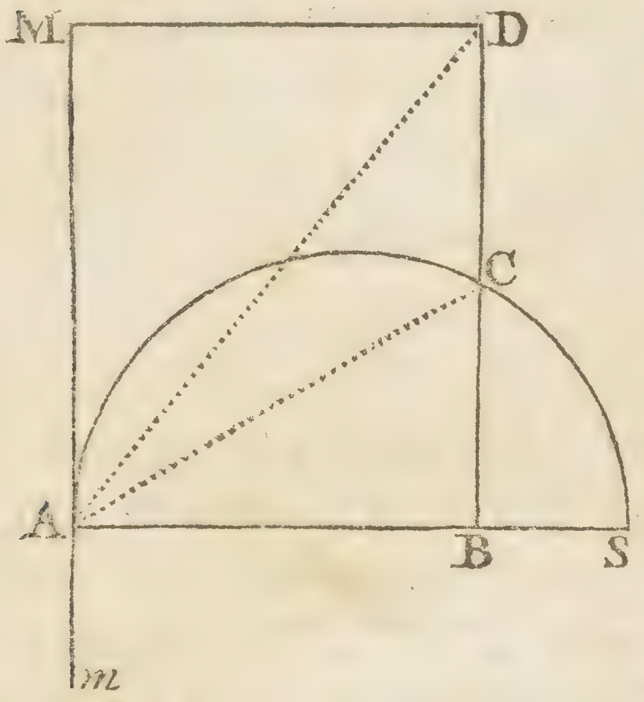
PRO-

PROBLEM XV.

110. To divide a given angle into three equal parts.

The Problem propofed contains three cafes; one is when the given angle is a right angle; another when it is obtufe; and the third when it is acute. The trifection of an angle.

Fig. 35.

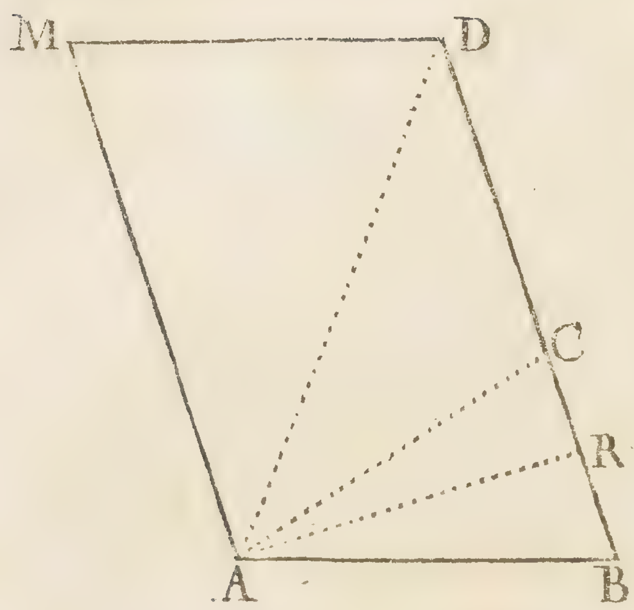


In the firft, let the given angle MAB be a right angle, which is fuppofed to be divided into three equal parts by the right lines AC, AD. Make AB $= a$, and at B raife the perpendicular BC, which produced fhall meet the line AD in D; and from the point D let DM be drawn parallel to AB. Then making BC $= x$, it will be AC $= \sqrt{aa + xx}$. But, becaufe the angle CAD muft be equal to the angle DAM, and becaufe of the parallels AM, BD, the angle DAM is equal to the angle ADC; the angles CDA, CAD, will be equal. Wherefore CD $=$ CA $= \sqrt{aa + xx}$, whence BD $= x + \sqrt{aa + xx}$. But befides, the two angles BAC, CAD, or CDA, ought alfo to be equal, and therefore in the two triangles BDA, CAB, the angle CAB will be equal to the angle BDA, and the right angle at B is common. Therefore alfo the third BCA $=$ BAD, and confequently the triangles are fimilar. Whence we fhall have AB . BC :: BD . AB; that is, $a \,.\, x :: x + \sqrt{aa + xx} \,.\, a$; and thence the equation $aa = xx + x\sqrt{aa + xx}$; and tranfpofing the term xx, and fquaring, it will be $aaxx + x^4 = a^4 - 2aaxx + x^4$, and finally, $3aaxx = a^4$, or $x = \pm \sqrt{\frac{1}{3}aa}$.

Produce AB to S, fo that it may be BS $= \frac{1}{3}$AB $= \frac{1}{3}a$. On the diameter AS let the femicircle ACS be defcribed; the ordinate BC will be $= \sqrt{\frac{1}{3}aa} = x$. Then draw AC to the point C, and take CD $=$ AC, drawing AD. The given angle will be then divided into three equal parts. For, whereas it is BC $= \sqrt{\frac{1}{3}aa}$, it will be AC $= \sqrt{\frac{4}{3}aa} =$ CD, and AD $= \sqrt{ABq + BDq} = \sqrt{aa + \frac{5}{3}aa + 2a\sqrt{\frac{4aa}{9}}} = 2a$. Therefore AD . AB :: $2a \,.\, a$ :: 2 . 1, and DC . CB :: $\sqrt{\frac{4}{3}aa} \,.\, \sqrt{\frac{1}{3}aa}$ :: 2 . 1; that is, in the very fame ratio as AD to AB. Wherefore, by *Euclid*, vi. 3, the angle BAC $=$ CAD; and, becaufe of CD $=$ CA, it will be alfo the angle CAD $=$ CDA $=$ DAM. The negative value, which is equal to the pofitive, would ferve for the divifion of the angle mAB.

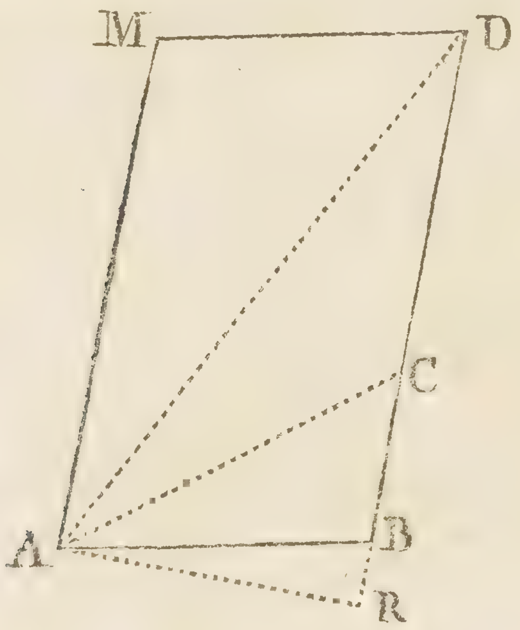
N Let

Fig. 36.



Let the angle BAM be obtuſe, and draw BD parallel to AM, and making the reſt as above, draw AR perpendicular to BD. Since the angle ABD is known, as being the ſupplement of the given angle MAB, and the angle R is right, and the line AB is given; the line BR will alſo be known, which make $= b$. Whence AR $= \sqrt{aa - bb}$, CR $= x - b$, AC $=$ CD $= \sqrt{aa - 2bx + xx}$, and BD $= x + \sqrt{aa - 2bx + xx}$. Then, becauſe of ſimilar triangles ABC, ABD, it will be AB . BC :: BD . BA; that is, $a \,.\, x :: x + \sqrt{aa - 2bx + xx} \,.\, a$; or $aa = xx + x\sqrt{aa - 2bx + xx}$. Then taking away the aſymmetry, it is $2bx^3 - 3aaxx + a^4 = 0$, which is a ſolid equation, or of the third degree, which at preſent I ſhall leave unreſolved.

Fig. 37.



Laſtly, let the angle BAM be acute; the perpendicular from the point A to DB produced will fall under the point B in R, and therefore it will be RC $= b + x$, and AC $= \sqrt{aa + 2bx + xx}$. Wherefore, repeating the ſame argumentation as in the foregoing caſe, we ſhall have the equation $2bx^3 + 3aaxx - a^4 = 0$, which differs from the foregoing only in the ſigns.

S E C T. III.

The Conſtruction of Loci, *or Geometrical Places, not exceeding the ſecond Degree.*

What are variable quantities; and what is the law by which they vary.

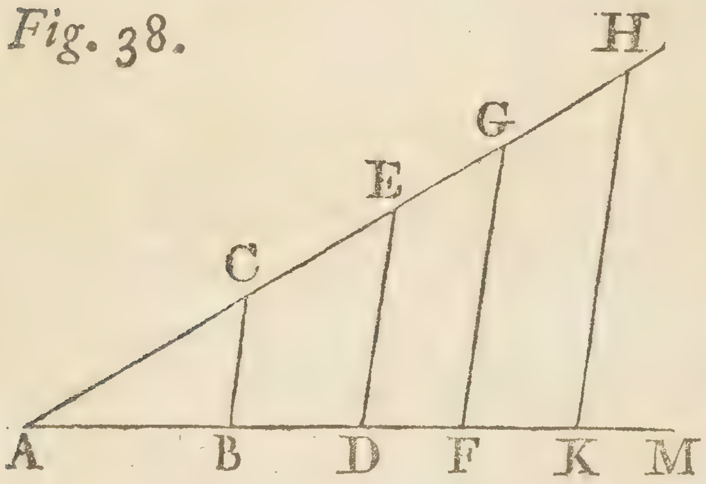
111. What are Indeterminate Problems, and how they require two unknown quantities, has been already explained at § 84. Now, becauſe the value of one of the unknown quantities may be varied an infinite number of ways, ſo, in like manner, the value of the other may be as often varied; whence they are called the *Variable Quantities* of the equation or problem, and their relation, or law which they obſerve in their variations, is expreſſed by an equation. Thus the equation

equation $bx = ay$ informs us, that, varying x as you pleaſe, y muſt alſo be varied, but with this condition, that x muſt always have to y the conſtant ratio of a to b. Thus the equation $ab = xy$ expreſſes ſuch a law, that the product of the two unknown quantities muſt always be conſtant, and equal to the product of a into b. The equation $ax = yy$ implies, that the ſquare of y muſt always be equal to the rectangle of x into a conſtant line a; and ſo of all other equations.

112. One of the two unknown quantities, ſuppoſe x for example, muſt have it's origin from a fixed point, and muſt be taken upon an indefinite right line. Then, if a determinate value be aſſigned to this, from the extremity is to be raiſed another right line in the given angle of the problem, which line is to be taken of ſuch a length as the other unknown line y ought to have, by the nature of the equation, relatively to the aſſigned or aſſumed value of x. And this ought to be repeated for every different value that x can aſſume. The line which ſhall paſs through the extremities of all the y's is called the *Locus* of the equation. The unknown line, which is taken from the fixed point on the indefinite right line, is called the *Abſciſs*; and the other, at the given angle to it, is called the *Ordinate:* and both indifferently are called the *Co-ordinates* of the equation.

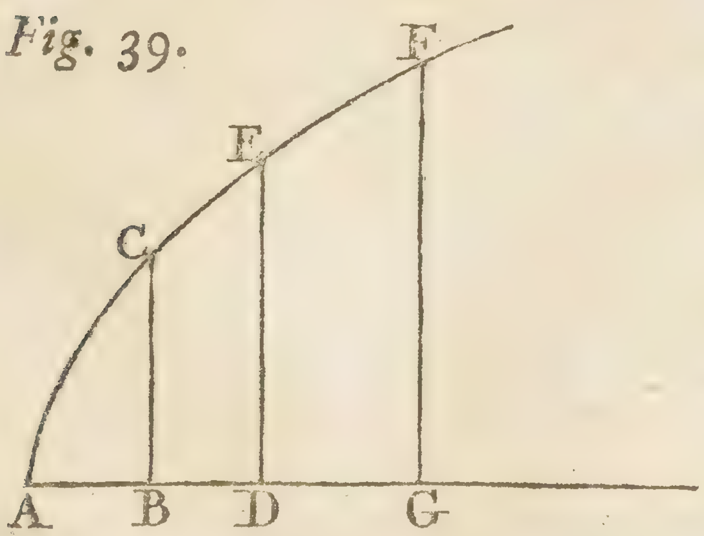
General precepts for the conſtruction of *Loci*, with ſome examples.

Fig. 38.



Now, for example, as to the equation $bx = ay$; upon the indefinite line AM take $AB = a$, and in any angle draw $BC = b$. Here, if we take $x = AD$, the fourth proportional will be parallel to BC, that is $DE = y$. And taking $x = AF$, then it will be $FG = y$. Alſo, taking $x = AK$, it will be $KH = y$. And thus for infinite others. And the line in which all theſe infinite points are found, C, E, G, H, &c. which are determined in this manner, will be the *locus* of the equation $bx = ay$, and which will be a right line.

Fig. 39.



In the ſame manner, as to the equation $ax = yy$, if we take $x = AB$, and $BC = \sqrt{ax}$, that is, a mean proportional between AB and the given line a, it will be $BC = y$. And taking $x = AD$, and DE a mean proportional between AD and a, it will be $DE = y$. Taking $x = AG$, and GF a mean proportional between AG and a, it will be $GF = y$. And ſo of all others. Now the points C, E, F, and infinite others determined in the ſame manner, will form the line ACEF, which is the *locus* of the equation $ax = yy$. And the ſame is to be underſtood of all other equations.

N 2 113. From

Different equations require different *loci*, and *vice versâ*.

113. From the feveral different laws expreffed by the given equations, or from the different relations that the two variables or unknown quantities may have to each other, other *loci* or lines will arife, which will differ both in kind and in degree. So it is eafy to perceive, that the *locus* of the equation $bx = ay$ will be a right line, as obferved before; for y to x having a conftant ratio, becaufe it is $y = \frac{bx}{a}$; any line ED (Fig. 38.) will be to AD, as any other FG to AF; therefore the triangles AED, AGF, will be fimilar. This may be verified alfo by any other point H, &c. So that it muft neceffarily follow, that thefe points will all be in the fame right line. But the equation $ax = yy$ requires, not that the lines BC, DE, &c. (Fig. 39.) but that their fquares, may have a conftant ratio to the correfponding lines AB, AD, &c. Whence it is, that the points C, E, F, &c. will not be in one right line, but in a certain curve line, called a *Parabola.* Thus a curve of a different kind from this would be the *locus* of the equation $xy = ab$; and a curve of a different kind and degree would be the *locus* of this other equation $a^3 - x^3 = y^3$. And the like of infinite others.

When the *locus* will be a right line.

114. As often as the equation fhall not contain, in any term, either the fquare, or fome higher power, of one of the unknown quantities, or the product of the fame, the *locus* will always be a right line.

When the *locus* is a conic fection.

115. And when, in the equation, there is found the fquare of one, or of the other, or of both the variable quantities, or their rectangle, either this or that as it may happen; and no term fhall include a greater power than the fquare of thofe variable quantities, or a product above the rectangle; that is, in no term the variable quantities, either alone or multiplied together, exceed the fecond dimenfion; the *locus* will always be one of the Conic Sections of *Apollonius*. Thefe affertions cannot be better demonftrated than by actually conftructing all the feveral equations of this nature.

Loci or curves diftinguifhed into orders.

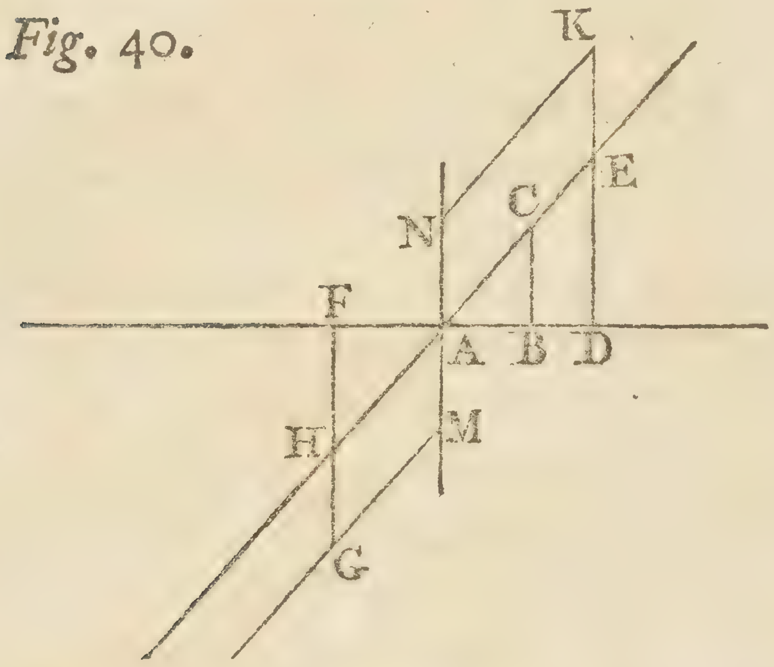
116. Equations which include the unknown quantities of one dimenfion only, that is, the *loci* to a right line, are called *Loci* or Lines of the Firft Order. Thofe which, either alone or multiplied together, include them of two dimenfions, that is, *loci* to the conic fections, are called *Loci* or Lines of the Second Order, and therefore Curves of the Firft Kind. Thofe equations in which the variables afcend to three dimenfions, are called *Loci* or Lines of the Third Order, and therefore Curves of the Second Kind. And fo on fucceffively.

The *loci* to a right line conftructed, in fix cafes.

117. Now, as to the *loci* to a right line, they are all comprehended under thefe fix equations following: $y = \frac{ax}{b}$, $y = -\frac{ax}{b}$, $y = \frac{ax}{b} + c$, $y =$

$-\frac{ax}{b}-c$, $y=\frac{ax}{b}-c$, and $y=-\frac{ax}{b}+c$. For, by multiplication and divifion, we may always reduce y to be free from fractions and co-efficients. By $\frac{a}{b}$ is to be underftood the aggregate of all the known quantities which multiply x, and by c the aggregate of all the quantities which form the given or conftant term.

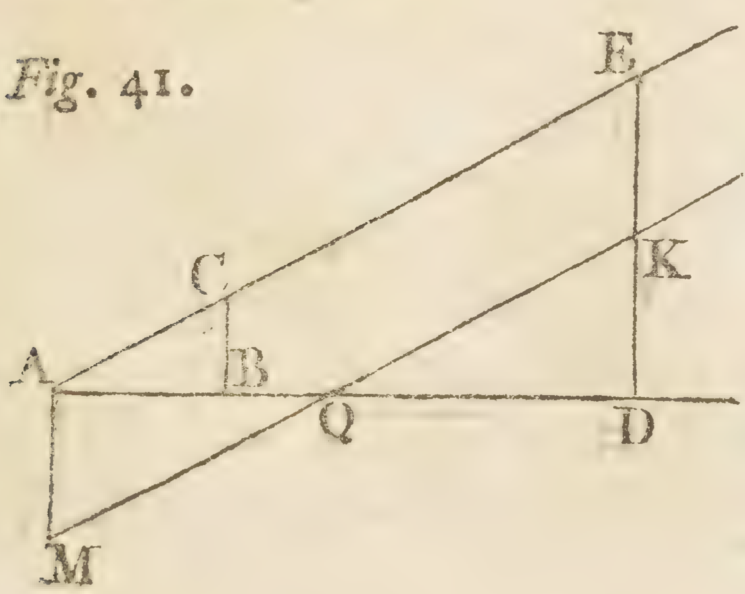
Fig. 40.



To conftruct the two firft, upon AD produced both ways indefinitely, take AB $=$ AF $=b$ on each fide, and draw BC $=a$, making the angle ABC fuch as the two variables of the problem ought to make. Through the points A, C, draw an indefinite right line HE; this will be the *locus* of the two equations $y=\frac{ax}{b}$, and $y=-\frac{ax}{b}$. For, taking any line AD $=x$, and drawing DE parallel to BC, it will be DE $=\frac{ax}{b}=y$. And taking AF $=-x$, and drawing FH parallel to BC, it will be FH $=-\frac{ax}{b}=y$.

The third and fourth are thus conftructed. Take AN $=$ AM $=c$, and parallel to BC; and draw NK, MG, indefinitely, and parallel to HE. NK will be the *locus* of the equation $y=\frac{ax}{b}+c$; and MG the *locus* of the equation $y=-\frac{ax}{b}-c$. For, taking AD $=x$, it will be DE $=\frac{ax}{b}$. But it is EK $=$ AN $=c$, making DK parallel to BC. Then DK $=\frac{ax}{b}+c=y$. And taking AF $=-x$, and drawing FG parallel to BC, it will be FG $=-\frac{ax}{b}-c=y$.

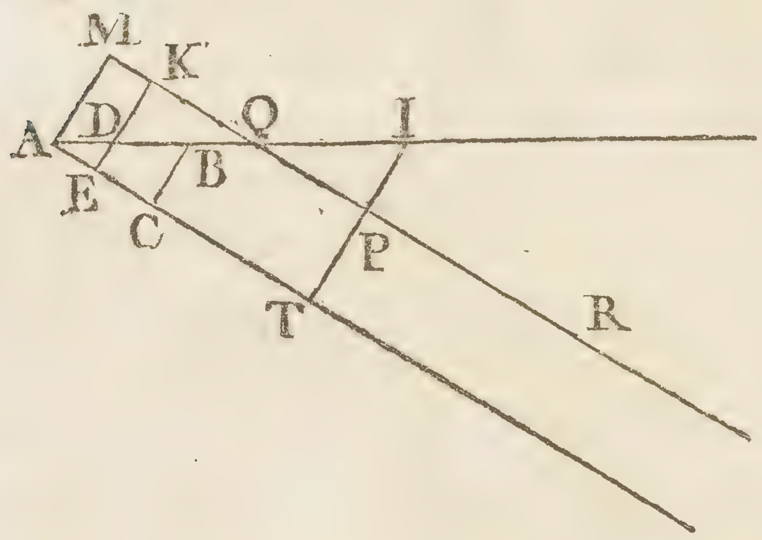
Fig. 41.



As to the fifth, conftruct the fame triangle ABC, and produce the lines AE, AD, indefinitely; draw AM $=c$, and parallel to BC. Then from the point M draw the indefinite line MK parallel to AE, which will meet the right line AD in Q. Then will QK be the *locus* of the equation $y=\frac{ax}{b}-c$. For, taking any line AD $=x$, and drawing DE parallel

parallel to BC, it will be DE $= \frac{ax}{b}$. But KE $=$ AM $= c$; therefore DK $= \frac{ax}{b} - c = y$. The portion QM will ferve when $\frac{ax}{b}$ is lefs than c, that is, when x is taken lefs than AQ, or lefs than $\frac{bc}{a}$; for, in this cafe, y will be negative, and therefore ought to be taken below AD, that is, the contrary way from DK.

Fig. 42.

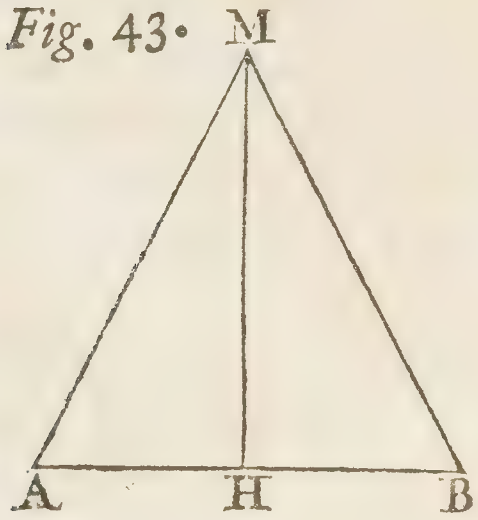


For the laft formula, make AB $= b$, BC $= a$, and the angle ABC equal to the fupplement of the angle of the variables. Make AM $= c$, parallel to BC, and draw MQK parallel to AC, cutting AB produced in Q. Then will MQK be the *locus* of the equation $y = c - \frac{ax}{b}$. For, taking any how AD $= x$, and drawing DE parallel to BC, it will be DE $= \frac{ax}{b}$. But, producing ED to K, it will be EK $=$ AM $= c$, and therefore DK $= c - \frac{ax}{b} = y$. Now, if x be taken greater than AQ, for inftance $=$ AI, it will be IT $= \frac{ax}{b}$, and therefore $c - \frac{ax}{b}$ is a negative quantity $= y =$ IP; taken directly contrary to DK, and the indefinite line MR is the *locus* of the propofed equation in both cafes.

The *locus* when one of the variables vanifhes.

118. It may fometimes happen, that, in the folution of a problem the *locus* of which is a right line, either one or the other of the two variables will difappear, and will not enter into the equation. In fuch cafes, the *locus* will be to the perpendicular, or to a parallel to the given right line upon which the abfciffes are taken, according as either the ordinate or abfcifs vanifhes. Here is an example or two of this.

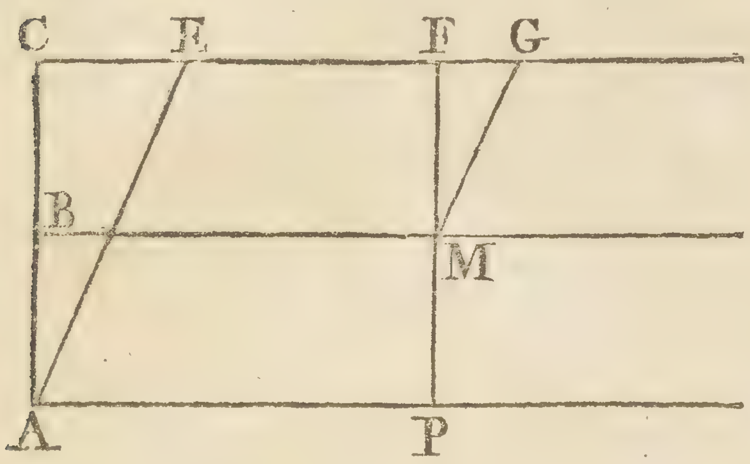
Fig. 43.



The right line AB being given, let it be propofed to find the *locus* of the points M out of this, fuch that, drawing the right lines MA, MB, to the extremities of AB, it may always be MA $=$ MB. Taking any line AH $= x$, draw HM $= y$, and make AB $= a$. It will be HB $= a - x$, AM $= \sqrt{xx + yy}$, and BM $= \sqrt{aa - 2ax + xx + yy}$; and thence the equation $\sqrt{xx + yy}$ $=$

$= \sqrt{aa - 2ax + xx + yy}$, and ſquaring, $xx + yy = aa - 2ax + xx + yy$, that is, $x = \frac{1}{2}a$; where y diſappears, and x remains determined. This ſhows us, that, taking $x =$ AH, which is half AB, and from the point H raiſing an indefinite perpendicular, every one of it's points will ſatisfy the queſtion, and therefore this will be the *locus* required.

Fig. 44.

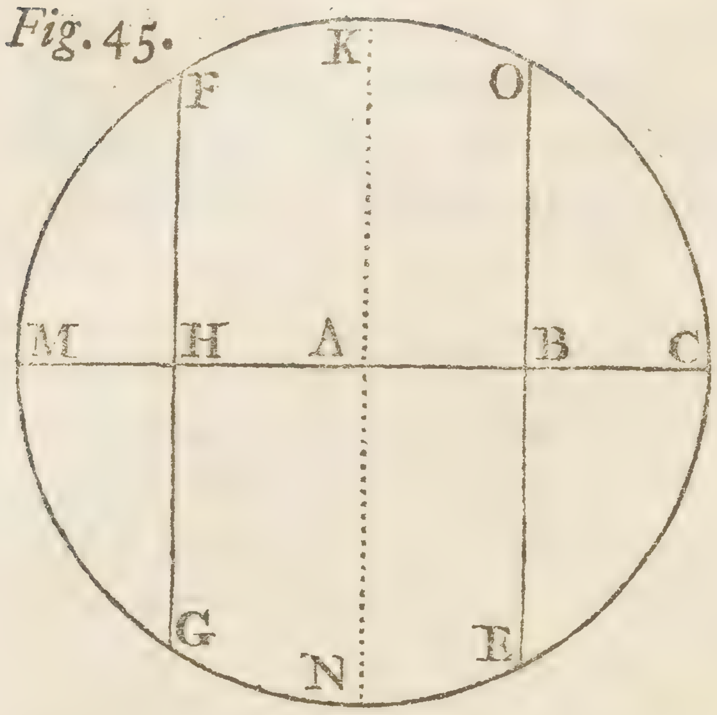


Let the parallels CG, AP, be given in poſition, and between them let it be required to find the *locus* of all the points M ſuch, that, drawing MP perpendicular to AP, and MG making the angle MGC equal to a given angle AEC; it may always be MP to MG in the conſtant ratio of a to b. Make the diſtance AC $= c$, AP $= x$, PM $= y$, and producing PM to F, it will be FM $= c - y$. Now, becauſe the angle AEC is given, and ACE is a right angle, and the ſide AC is given, the ſide AE will alſo be known, which may be called f. Now, becauſe of the ſimilar triangles ACE, FMG, it will be AC . AE :: MF . MG; that is, $c . f :: c - y . \text{MG} = \frac{cf - fy}{c}$. But beſides, it ought to be PM . MG :: $a . b$. Then it will be $y . \frac{cf - fy}{c} :: a . b$, and therefore $bcy = acf - afy$, or $y = \frac{acf}{bc + af}$. So that here is an equation, in which the unknown quantity x does not enter at all. Therefore, taking x as you pleaſe, y will always be conſtant, and equal to $\frac{acf}{bc + af}$; and therefore, drawing the indefinite line BM parallel to AP, and as far diſtant from it as the quantity $\frac{acf}{bc + af}$, this line will be the *locus* required.

119. Having thus explained the conſtruction of the *Loci* to a Right Line, I come now to the conſtruction of Equations of the Second Degree, or of the *Loci* to the Conic Sections. And here I muſt ſuppoſe the learner to be ſo well inſtructed in the chief geometrical properties of theſe ſections of the cone, as to form from thence the firſt and more ſimple equations of theſe curves; to which ſimple equations the more compounded ones may be reduced and referred, by the methods now to be explained. The *loci* to a circle conſtructed.

And, in the firſt place, it muſt be known, that in the circle any ordinate is a mean proportional between the ſegments of the diameter; that is, it's ſquare is equal to the rectangle of the ſaid ſegments. Therefore, in the circle MKCN, if

Fig. 45.



if you make the radius AC $= a$, and from the centre A any abſciſs whatever AB $= x$, and the perpendicular ordinate BD $= y$, it will be MB $= a + x$, BC $= a - x$, and therefore MB $\times$ BC $= aa - xx$; then it will be $yy = aa - xx$, an equation to the circle, in reſpect of the quadrant KC. But, becauſe the ſame property may be verified alſo, taking BE for the ordinate, that is the negative ordinate $- y$, and as well the ſquare of $- y$ as of y is yy; therefore the ſame equation belongs alſo to the quadrant CN. And now, if we take the abſciſſes negative, as AH $= - x$, and the ordinates HF $= y$, HG $= - y$, their ſquare yy will, in both caſes, be equal to the rectangle MH $\times$ HC. But when it is AH $= - x$, it will be CH $=$ CA $+$ AH $= a - x$; and MH $=$ AM $-$ AH $= a + x$ by the rules of Addition and Subtraction. And therefore the rectangle MH $\times$ HC will be ſtill $aa - xx$. So that $yy = aa - xx$ is the moſt ſimple equation that belongs to the whole circle with radius a, taking the abſciſſes from the centre.

If the abſciſſes ſhould be taken, not from the centre A, but from M the extremity of the diameter, making any one of them MH or MB equal to x, it will be HC or BC $= 2a - x$, and the rectangle of the ſegments will be equal to $2ax - xx$. But the ſquare of the ordinate, as well poſitive as negative, is yy, ſo that it will be $yy = 2ax - xx$; the moſt ſimple equation of the ſame circle, taking the abſciſſes not from the centre, but from the extremity of the diameter.

By the quantity or magnitude a, which denotes the radius, is meant any given quantity whatever, whether ſimple or compound, integer or fraction, rational or ſurd; ſo that $yy = aa - bb - xx$ will be a circle with radius $= \sqrt{aa - bb}$; $yy = \frac{aab}{m} - xx$ will be a circle with radius $= \sqrt{\frac{aab}{m}}$; $yy = a\sqrt{ab} - xx$ will be a circle with radius $= \sqrt{a\sqrt{ab}}$. Thus $yy = 2ax - bx - xx$ will be a circle with diameter $= 2a - b$, or with radius $= \frac{2a - b}{2}$; $yy = \frac{aax + abx}{b} - xx$ will be a circle with diameter $= \frac{aa + ab}{b}$; $yy = x\sqrt{ab} - xx$ will be a circle with diameter $= \sqrt{ab}$. And ſo of others.

Here it is plain, that, in the equation $yy = aa - bb - xx$, and in all others like it, if the quantity b ſhould be greater than a; then $aa - bb$ being a negative quantity, the circle would become imaginary. For then the ordinate

 $y =$

$y = \sqrt{aa - bb - xx}$ being equal to the ſquare-root of a negative quantity, it would be therefore imaginary.

For the ſame reaſon, in the equation $yy = 2ax - xx$, the abſciſs x cannot be taken negative; for, taking x negative, the term $2ax$ would be negative, and therefore the equation $yy = -2ax - xx$, that is $y = \sqrt{-2ax - xx}$, would be an imaginary quantity.

120. The primary property of the *Apollonian* Parabola is this, that the ſquare of any ordinate whatever is equal to the rectangle of the parameter into the abſciſs; taken on the axis if the angle of the co-ordinates be a right angle, or on a diameter if that angle be oblique. Then, making the parameter $= a$, any abſciſs $AB = x$, the correſponding poſitive ordinate $BC = y$, and the negative $BD = -y$; then yy will be the ſquare as well of BC as of BD, and ax will be the rectangle of the parameter into AB. Wherefore $yy = ax$ is the moſt ſimple equation which belongs to the parabola with the parameter a. And here it is plain, that the abſciſs x cannot be taken negative, becauſe of the avoiding imaginary quantities. And here alſo, by the quantity a, which expreſſes the parameter, is to be underſtood any given quantity, into which the abſciſs x is multiplied; ſo that $\frac{aax \pm bbx}{c} = yy$ will be a parabola, the parameter of which is $= \frac{aa \pm bb}{c}$. And $x\sqrt{ab} = yy$ will be a parabola, the parameter of which is $\sqrt{ab}$. And the like of all others. The ſimpleſt *loci* to the parabola conſtructed.

Fig. 46.

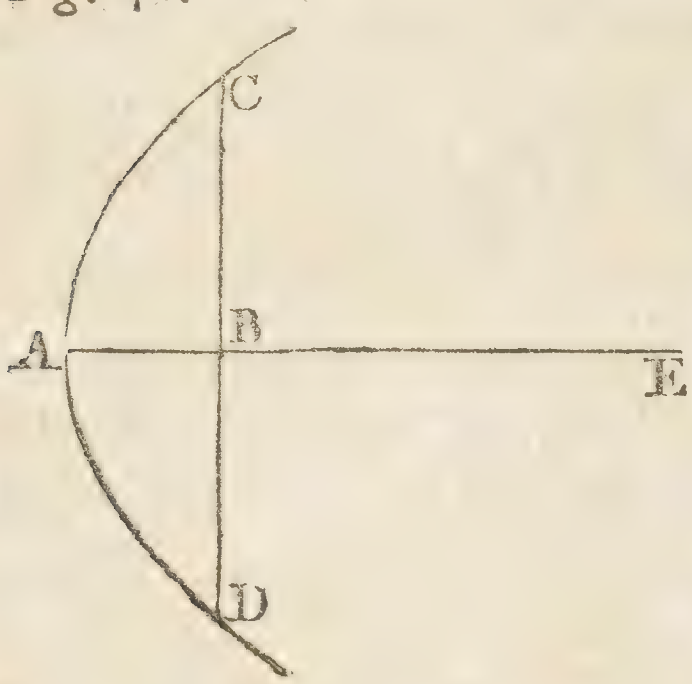
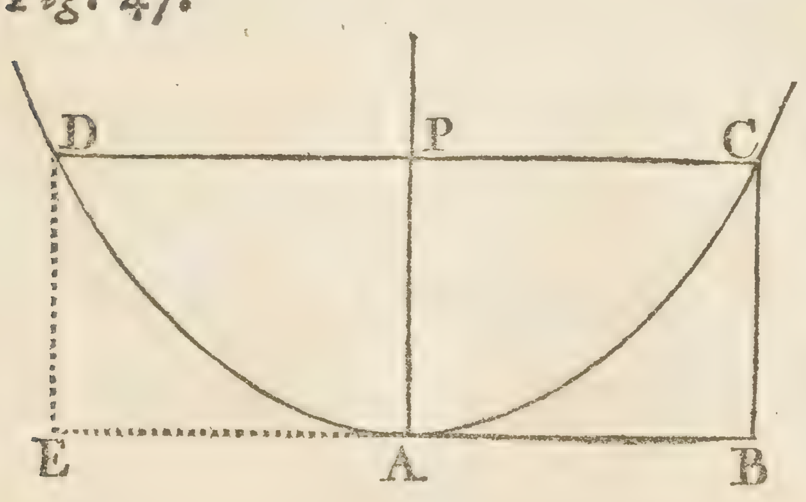


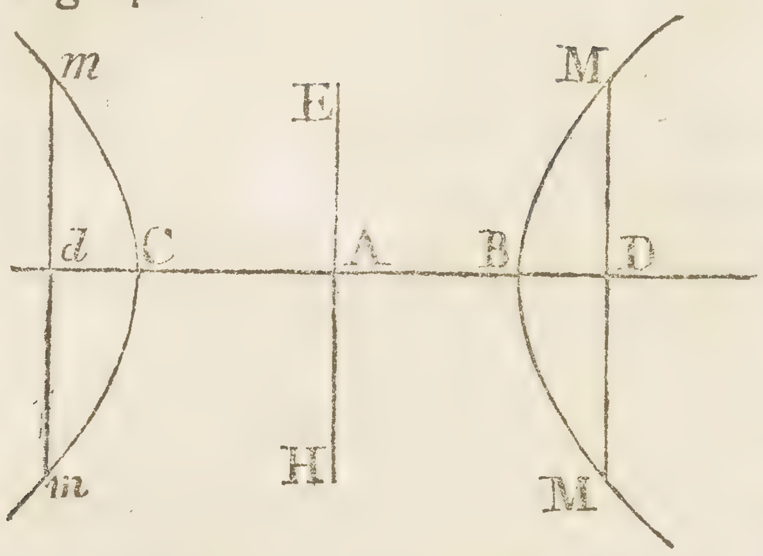
Fig. 47.



If the parabola ſhould be differently placed, as in Fig. 47, and on the ſame line AB, from the given point A, we ſhould take the abſciſſes, or x; the equation would be $xx = ay$, in which we may take the abſciſs either poſitive or negative, but the ordinates muſt always be poſitive.

The simplest *loci* to the hyperbola constructed.

Fig. 48.



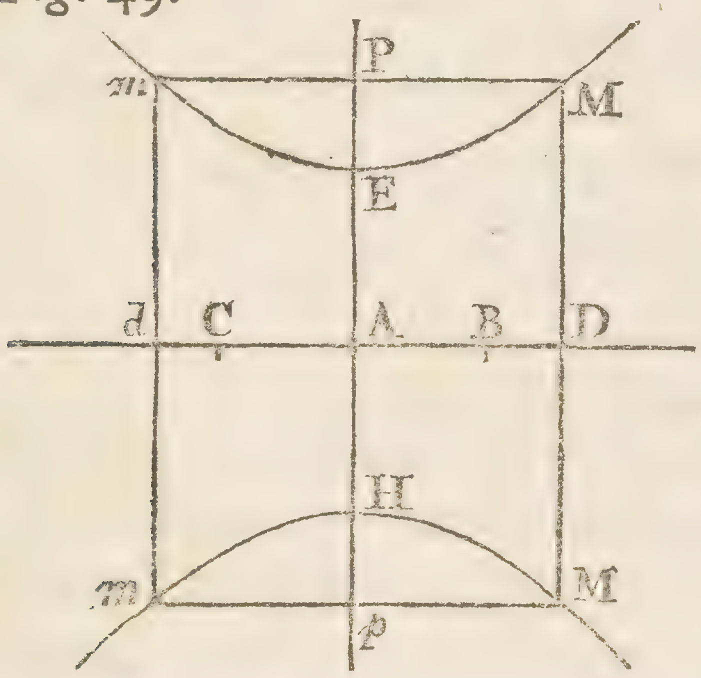
121. Let the opposite hyperbolas be referred to their axis, or to a diameter, according as the angle of the co-ordinates is either right or oblique; and let CB be the axis, or the transverse diameter, and HE the conjugate. By the known property of the hyperbola, taking D any point whatever, and drawing DM parallel to HE, the rectangle $CD \times DB$ must be to the square of DM, as the square of CB is to the square of HE. Then, making $CB = 2a$, $HE = 2b$, and from the centre A taking any line $AD = x$, DM positive $= y$, DM negative $= -y$, it will be $CD = a + x$, $BD = x - a$, and therefore, by the said property, $xx - aa \,.\, yy :: 4aa \,.\, 4bb$, that is, $xx - aa = \frac{aayy}{bb}$. And, taking Ad negative $= -x$, and the ordinates as before, it will be $Bd = -x + a$, $Cd = -x - a$, and the rectangle $Bd \times dC = xx - aa$. Whence, in the same manner, we shall have $\frac{aayy}{bb} = xx - aa$; the most simple equation expressing the two entire opposite hyperbolas referred to their axes or diameters, taking the abscisses from the centre. And, if we shall take the abscisses from the vertex C, we shall have the analogy (by the said property) $x \times \overline{x - 2a} \,.\, yy :: 4aa \,.\, 4bb$; that is, the equation $-2ax + xx = \frac{aayy}{bb}$. And lastly, taking the abscisses from the vertex B, we shall have $x \times \overline{2a + x} \,.\, yy :: 4aa \,.\, 4bb$; and therefore the equation $2ax + xx = \frac{aayy}{bb}$.

It is also a primary property of the opposite hyperbolas, that the same rectangle $CD \times DB$, taking the abscisses positive, and $Bd \times dC$, taking the abscisses negative, is to the square of the ordinate, whether positive or negative, as the axis or transverse diameter is to the parameter. Making, therefore, the parameter $= p$, and other things as before, it will be $xx - aa \,.\, yy :: 2a \,.\, p$; that is, $\frac{2ayy}{p} = xx - aa$; the most simple equation expressing the two opposite hyperbolas as referred to a parameter, and taking the abscisses from the centre. Now, taking the abscis from the vertex C, the equation will be $\frac{2ayy}{p} = xx - 2ax$; and lastly, taking the abscis from the vertex B, the equation will be $2ax + xx = \frac{2ayy}{p}$.

If

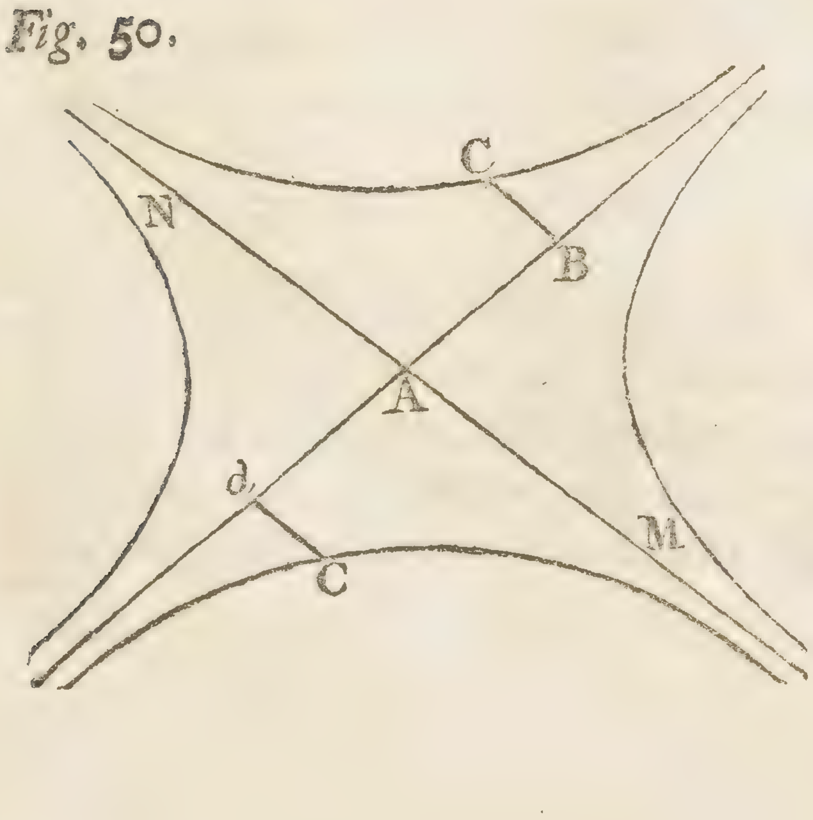
If the hyperbolas be equilateral, becauſe, in this caſe, the two axes or diameters are equal to each other, and equal to the parameter, each equation will become $yy = xx - aa$, taking the abſciſs from the centre; or $yy = 2ax + xx$, taking the abſciſs from the vertex B; or $yy = -2ax + xx$, taking the abſciſs from the vertex C. By the quantity aa is to be underſtood any plane however complicated, as alſo by the quantity bb. And by $2a$, as alſo by p, is underſtood any line whatever. So that, in the equation $\frac{\overline{aa + ff} \times yy}{b\sqrt{ab}} = xx - aa - ff$, we ſhall have $\sqrt{aa+ff}$ for the ſemiaxis, or tranſverſe ſemidiameter, and $2\sqrt{aa+ff}$ will be the whole axis or diameter. As alſo, $\sqrt{b\sqrt{ab}}$ is the ſemiaxis or ſemidiameter conjugate, and $2\sqrt{b\sqrt{ab}}$ is the whole. In the equation $\frac{a^3yy}{bbc} = xx - \frac{a^3}{c}$, it will be $\sqrt{\frac{a^3}{c}}$, the ſemiaxis or tranſverſe ſemidiameter, and b the conjugate. In the equation $xx - bx = \frac{byy}{c+m}$, it will be b the ſemiaxis or tranſverſe ſemidiameter, and $c + m$ the parameter. In the equation $\frac{2yy\sqrt{aa-bb}}{a-b} = xx - aa + bb$, it will be $2\sqrt{aa-bb}$ the axis or tranſverſe diameter, and $a - b$ the parameter. And ſo on.

Fig. 49.



If the oppoſite hyperbolas ſhall be differently ſituated, as in Fig. 49, and upon the ſame diameter CB equal to $2a$, produced, if you would have the x's poſitive, and negative from the centre A, (it being HE $= 2b$,) the equation would be $yy - bb = \frac{bbxx}{aa}$.

Fig. 50.



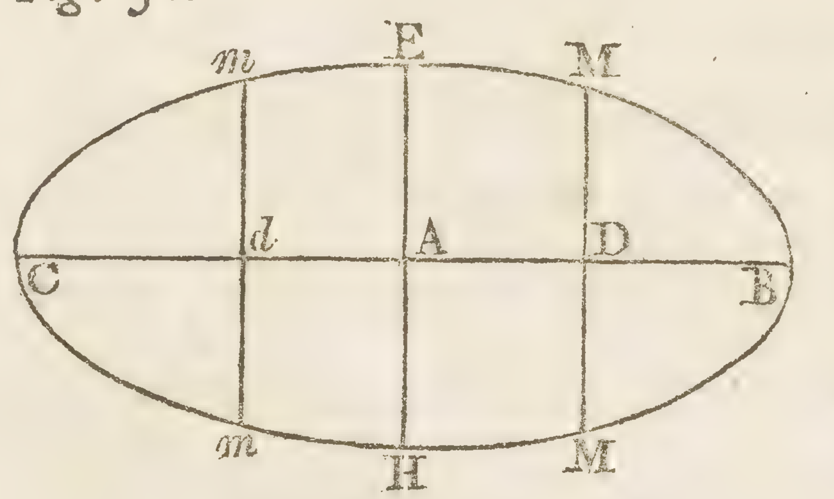
122. In the hyperbola between the aſymptotes, the rectangle of any line AB taken on the aſymptote dB, into the ordinate BC parallel to the aſymptote MN, or A$d \times d$C, is always conſtant, that is, equal to a known rectangle. Therefore, making AB $= x$, BC $= y$, and the known rectangle $= ab$, it will be $xy = ab$; and, taking Ad negative $= -x$, and dC negative $= -y$, the rectangle A$d \times d$C ſhall be

The ſimpleſt *loci* of the hyperbola between it's aſymptotes conſtructed.

be alſo xy; and therefore $xy = ab$ is the moſt ſimple equation belonging to the oppoſite hyperbolas between the aſymptotes. It is plain, that the equation $-xy = ab$, or $xy = -ab$, will ſerve for the oppoſite hyperbolas in the angles BAM, bAN, one of the co-ordinates being always poſitive, and the other negative, and therefore the product is negative.

The ſimpleſt *loci* to the ellipſis conſtructed.

Fig. 51.



123. In the ellipſis CEBH, taking from the centre A any line AD upon the axis or tranſverſe diameter CB, and drawing DM parallel to the axis or conjugate diameter EH; by the known property of the ellipſis, the rectangle CD $\times$ DB muſt be to the ſquare of DM, as the ſquare of the axis or tranſverſe diameter CB is to the ſquare of the conjugate HE. Therefore, making CB $= 2a$, HE $= 2b$, and from the centre A taking any line AD $= x$, and making DM poſitive $= y$, DM negative $= -y$; it will be CD $= a + x$, DB $= a - x$, and therefore $aa - xx \,.\, yy :: 4aa \,.\, 4bb$; that is, $\frac{aayy}{bb} = aa - xx$. And taking A$d$ negative $= -x$, and the ordinates as before, it will be Bd = BA + Ad $= a - x$, dC = AC − Ad $= a + x$, and therefore the rectangle B$d \times d$C ſhall be alſo $= aa - xx$. Whence, in the ſame manner, we ſhall have $aa - xx = \frac{aayy}{bb}$, the moſt ſimple equation to the ellipſis, taking the abſciſſes from the centre. And if we ſhould take the abſciſſes from the vertex C, we ſhould have the analogy $2ax - xx \,.\, yy :: 4aa \,.\, 4bb$; and therefore the equation $\frac{aayy}{bb} = 2ax - xx$.

It is alſo a known property of the ellipſis, that the ſame rectangles are to the ſquares of the correſpondent ordinates, as the axis or tranſverſe diameter is to the parameter. Therefore, calling this parameter p, and every thing continuing as before, it will be $aa - xx \,.\, yy :: 2a \,.\, p$. Therefore it is $\frac{2ayy}{p} = aa - xx$, the moſt ſimple equation of the ellipſis referred to it's parameter, taking the abſciſſes from the centre. And, taking the abſciſſes from the vertex C, the equation of the ellipſis referred to it's parameter will be $\frac{2ayy}{p} = 2ax - xx$.

If the two axes ſhall be equal to each other, in which caſe they are alſo equal to the parameter, both of the equations will become $yy = aa - xx$, taking the abſciſſes from the centre; and $2ax - xx = yy$, taking the abſciſſes from the point C. But, if we confine it to an axis in which the angle of the co-ordinates is a right angle, the ellipſis will degenerate into a circle with radius $= a$.

The obſervation made in the hyperbola, concerning the given quantities aa, bb, $2a$, p, in reſpect to the diameters and parameter, is to be underſtood equally of the ellipſis, to ſave needleſs repetitions.

124. Now, in equations belonging to the hyperbola and the ellipſis, as referred to the axis or diameters, taking the abſciſs from the centre; as

In theſe *loci* the diameters may be found, if not given.

$$\frac{aayy}{bb} = xx - aa, \quad \frac{aayy}{bb} = aa - xx,$$

the ſquare-root of the conſtant term, or of aa, will always be the tranſverſe ſemiaxis or ſemidiameter. And if the co-efficient of the ſquare of the ordinate be the ſame conſtant term divided by any given quantity, the root of this diviſor is always the conjugate ſemiaxis or ſemidiameter, that is, the root of bb. But if this co-efficient be not ſuch, or do not contain the conſtant term after this manner, then the ſemiaxis or conjugate ſemidiameter will be different. Thus, for example, in the equation $\frac{ffyy}{bb} = xx - aa$, the ſemiaxis, or half the tranſverſe diameter, is indeed always a, but b is not the conjugate. To find this we muſt make an analogy: As the numerator of the co-efficient of the ſquare of the ordinate is to it's denominator, ſo is the conſtant term to a fourth, the root of which will be the ſemiaxis or ſemidiameter required. Then, in equations to the ellipſis or hyperbola referred to the axis or diameter, taking the abſciſs from the vertex, as in $\frac{aayy}{bb} = 2ax - xx$, $\frac{aayy}{bb} = xx - 2ax$, $\frac{aayy}{bb} = xx + 2ax$, the tranſverſe ſemiaxis or ſemidiameter ſhall be half of that quantity, which multiplies the unknown quantity in it's firſt dimenſion, and the conjugate as before. Obſerving, that when the co-efficient of the ſquare of the ordinate is not the ſquare of the axis or tranſverſe diameter thus found, the analogy for the ſemiaxis or conjugate ſemidiameter will be thus: As the numerator of the co-efficient of the ſquare of the ordinate is to the denominator, ſo the ſquare of half the quantity that multiplies the unknown quantity of the firſt dimenſion, is to a fourth; and the ſquare-root of this fourth proportional ſhall be the conjugate ſemiaxis or ſemidiameter.

Therefore, in the equation to the hyperbola $\frac{ffyy}{bb} = xx - aa$, the tranſverſe ſemiaxis or ſemidiameter will be $= a$, and the conjugate $= \frac{ab}{f}$. And ſince, by the property of the curve, it ought to be: As the rectangle of the ſum into the difference, (of the tranſverſe ſemiaxis or ſemidiameter and the abſciſs,) is to the ſquare of the ordinate, ſo is the ſquare of the axis or tranſverſe diameter to the ſquare of the conjugate; it will be $xx - aa \,.\, yy :: 4aa \,.\, \frac{4aabb}{ff}$, or $\frac{4aayy}{4aabb} \times ff = xx - aa$, that is, $\frac{ffyy}{bb} = xx - aa$, which is the propoſed equation.

Thus,

Thus, in the equation $\frac{abyy}{cc} = xx - aa$, the tranfverfe femiaxis or femidiameter $= a$, and the conjugate $= \sqrt{\frac{acc}{b}}$. In the equation $xx - 2ax = \frac{bbyy}{cm}$, the tranfverfe femiaxis or femidiameter $= a$, and the conjugate $= \frac{a}{b}\sqrt{cm}$. In the equation $\frac{aa - bb}{cc}yy = xx - bb$, the tranfverfe femiaxis or femidiameter will be $= b$, and the conjugate $= \sqrt{\frac{bbcc}{aa - bb}}$, &c.

To find the *loci* when referred to a parameter.

125. If the equations be referred to parameters, as $\frac{2ayy}{p} = aa - xx$, or $\frac{2ayy}{p} = xx - aa$, taking the abfciffes from the centre; or $\frac{2ayy}{p} = 2ax - xx$, or $\frac{2ayy}{p} = 2ax + xx$, or $\frac{2ayy}{p} = xx - 2ax$, taking the abfciffes from the vertex; in the firft, the tranfverfe femiaxis or femidiameter will always be the root of the conftant term; and in the fecond, the half of the co-efficient of the unknown quantity of the firft dimenfion; and the parameter will always be the quantity of the denominator of the co-efficient of the fquare of the ordinate, when the numerator of the fame co-efficient in the firft is double to the root of the conftant term; and in the fecond, is equal to the quantity which multiplies the unknown quantity of the firft dimenfion. But when the faid denominator has not the afore-mentioned conditions, the parameter fhall be the fourth proportional to the numerator, the denominator, and the axis or tranfverfe diameter.

Therefore, in the equation to the ellipfis $aa - xx = \frac{byy}{c}$, the axis or tranfverfe diameter fhall be $= 2a$, and the parameter $= \frac{2ac}{b}$. And, fince it ought to be, by the property of the ellipfis, as the rectangle of the fum into the difference of the femiaxis or tranfverfe femidiameter and the abfcifs, is to the fquare of the ordinate, fo the axis or tranfverfe diameter is to the parameter; it will be $aa - xx \,.\, yy :: 2a \,.\, \frac{2ac}{b}$, that is, $\frac{byy}{c} = aa - xx$, which is the equation propofed. In the equation $xx - aa = \frac{3yy}{4}$, which is to the hyperbola, the axis or tranfverfe diameter $= 2a$, the parameter $= \frac{8a}{3}$. In the equation to the hyperbola $2ax + xx = \frac{b - c}{m}yy$, the axis or tranfverfe diameter will be $2a$, and the parameter $\frac{2am}{b - c}$. In the equation to the ellipfis $aa - bb$

— xx

$-xx = \frac{byy}{c}$, the axis or tranſverſe diameter will be $= 2\sqrt{aa-bb}$, and the parameter $= \frac{2c\sqrt{aa-bb}}{b}$; ſuppoſing a to be greater than b, for otherwiſe the curve would be imaginary.

126. Theſe things being premiſed, and well underſtood, the conſtruction of more complicate equations, or of all other *Loci* to the conic ſections, will be very eaſy; and that by reducing ſuch complicate equations to the ſimple primary equations here exhibited. So that, the deſcription of ſuch a conic ſection being ſuppoſed, we may proceed to the conſtruction of the propoſed equation. The *loci* to the conic ſections diſtributed into three ſpecies.

Now, to proceed with the greater perſpicuity, I ſhall diſtribute all equations to the conic ſections into three ſpecies or claſſes, I mean all complicate ones. Thoſe of the firſt claſs ſhall be all ſuch as contain the ſquare of only one of the unknown quantities, and the rectangle of the other unknown quantity into a conſtant quantity. As, for example, $ax \pm ab = yy$. And moreover, all thoſe ſhall be ſaid to be of the firſt ſpecies, which contain rectangles of the unknown quantities one among another, and with conſtant quantities, but have not the ſquare of either of the unknown quantities. As $xy + ax = aa - ay$; the ſigns being of any kind, which is alſo to be underſtood of the ſigns of the other two ſpecies.

Of the ſecond ſpecies I call thoſe, in which there are the ſquares of one or both the unknown quantities, and alſo their rectangles into conſtant quantities, but not their rectangle into each other; as $xx + 2ax = ay + by$, or $xx - 2bx = yy + ay - ax$.

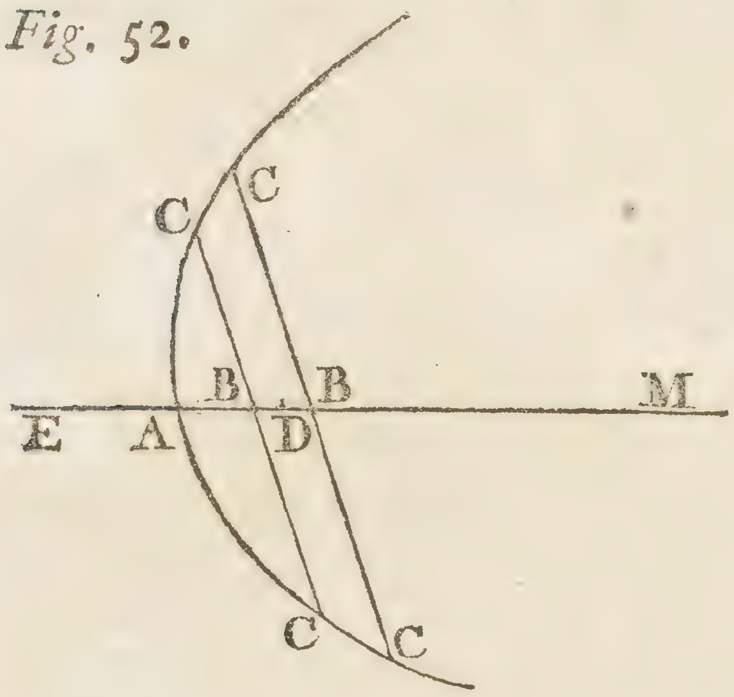
Thoſe are of the third ſpecies, in which are contained rectangles of the two unknown quantities into each other, and other terms of what kind ſoever; ſuch as $xx + 2xy + 2yy = aa - xx + bx$.

127. To diſtinguiſh and conſtruct equations of the firſt ſpecies, there is occaſion to make uſe of one ſubſtitution, which is, to put the unknown quantity which has no ſquare, *plus* or *minus* (according to the ſigns), a conſtant quantity, equal to ſome new unknown quantity; and thus to reduce the equation, (repeating this ſubſtitution if there be occaſion,) to a more ſimple expreſſion, ſo that the *locus* of the ſaid equation may be eaſily known and conſtructed; as may be ſeen in the following Examples. *Loci* of the firſt ſpecies conſtructed, with examples.

EXAMPLE I.

Let the equation be $ax + ab = yy$, and let the angle be given, which the co-ordinates make with each other. Becaufe $ax + ab$ is the fame as $a \times \overline{x + b}$, make $x + b = z$; then, by fubftitution, it will be $az = yy$, which is the *Apollonian* parabola.

Fig. 52.



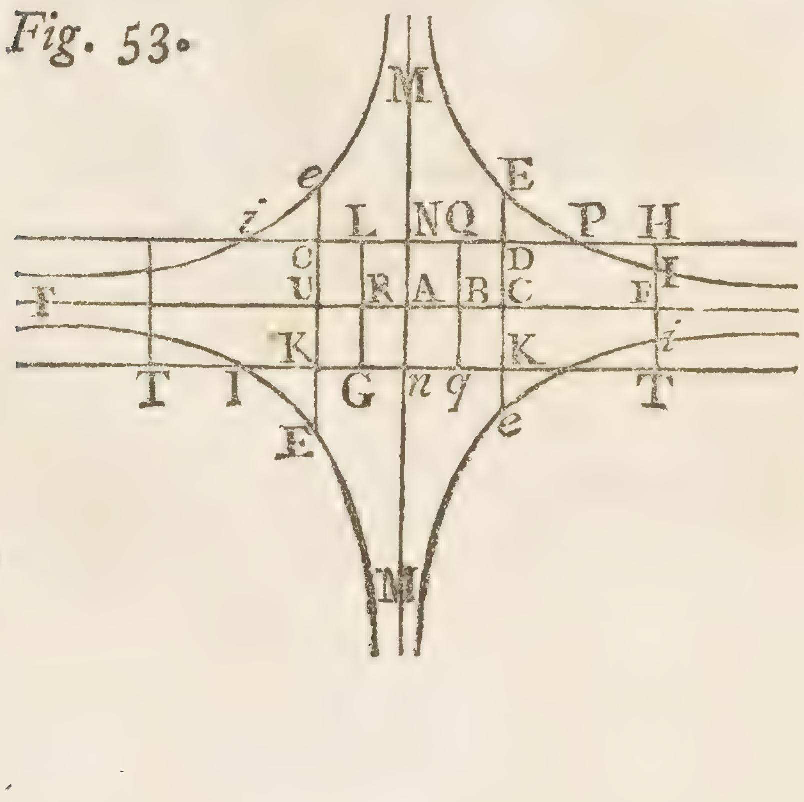
On the indefinite line AB as a diameter, with a parameter $= a$, let the parabola CAC be defcribed, whofe co-ordinates AB, BC, contain the given angle; then let AD $= b$. Taking any line AB $= z$, it will be BC $= y$. But, becaufe, by the fubftitution, we have $x = z - b$, DB will be x. Therefore the origin of the abfcifs x will be the point D, taking the pofitive towards M, the negative towards A, and the correfponding pofitive and negative ordinates will be y.

If the propofed equation had been put $ax - ab = yy$, we fhould have made the fubftitution $x - b = z$, and therefore $x = z + b$. In which cafe, taking AE $= b$ in the diameter produced, and doing the reft as before, the point E would then have been the origin of the abfcifs x.

EXAMPLE II.

Let the equation be $xy + ax = aa - ay$. Make $y + a = z$, and, inftead of y, fubftituting this value $z - a$, we fhall have $zx + az = 2aa$; and making another fubftitution of $x + a = p$, it will be $pz = 2aa$, the *Apollonian* hyperbola between the afymptotes.

Fig. 53.



Let the indefinite right lines MM, FF, comprehend the given angle of the co-ordinates, and between the afymptotes MM, FF, let the two oppofite hyperbolas be defcribed, belonging to the conftant rectangle $2aa$. Taking any line AC $= p$, and the ordinate CE parallel to AM, it will be $= z$. But, by the fubftitution, it is $x = p - a$; therefore, making AB$=a$, it will be BC$=x$. And,

And, becaufe we have alfo, by the other fubftitution, $y = z - a$, making $AN = a$, and drawing NH parallel to FF, it will be $DE = y$. Therefore, drawing BQ parallel to AN, Q will be the beginning of the abfcifs x. Thus, to any abfcifs $QD = x$ will correfpond the ordinate $DE = y$, pofitive between the points Q and P, and negative beyond the point P, as HI. But, when p is taken lefs than a, that is, AC lefs than AB, then, as it is $x = p - a$, x will be negative, that is, towards N; and to it will correfpond the pofitive ordinates y. Now, if we take p negative, and equal to AU for example, x will be negative, and equal to QO, and y negative $=$ OE. If the equation were $xy + ax = aa + ay$, or elfe, $xy + ax = - aa - ay$, or this, $xy - ax = aa - ay$, or this, $xy - ax = - aa + ay$; the two firft would be divifible by $y + a$, and we fhould have $x = \pm a$. The two others would be divifible by $y - a$, and we fhould have $x = \pm a$. Therefore they would not be *loci*, but equations of determinate problems. But if it were $xy - ax = aa + ay$, the firft fubftitution would be $y - a = z$, whence the equation $zx - az = 2aa$; and confequently the fecond fubftitution would be $x - a = p$; whence finally the equation $zp = 2aa$; and therefore, in this cafe, to the co-ordinates p, z, muft be added the quantity a, in order to have x and y. And therefore, taking from A towards U the line $AR = a$, and drawing RG parallel to MN and equal to a, then, through the point G drawing GT parallel to FF, G fhall be the origin of the abfciffes x, and the correfponding ordinates fhall be y.

If the equation were $xy + ax = - aa + ay$, the fubftitutions would be $y + a = z$, and $x - a = p$, which would give us the equation $pz = - 2aa$.

Let the fame hyperbolas be defcribed, but in the other two angles, becaufe the conftant rectangle $2aa$ is negative; and let them be *ie*, *ie*. Producing GR to L, this will be the origin of x both affirmative and negative. And upon the right line LQ, produced both ways, the ordinates y will infift, that is, negative from N towards H, and pofitive from N to the point i; and again negative beyond the point i.

If it were $xy - ax = - aa - ay$, the fubftitutions would be $y - a = z$, and $x + a = p$. Therefore, the fame hyperbolas *ie* being defcribed, and QB being produced to q, this will be the origin of the abfciffes x, and the ordinates y will infift upon TT.

If, in the equations, the term xy fhould be negative, it may be made pofitive by tranfpofing the terms.

The diverfity of fubftitutions, and of the pofition of the co-ordinates, which arifes from the different combinations of the figns in the propofed equations, and whatever elfe has been confidered here, is to be fupplied in what follows, where, for brevity-fake, I fhall omit it.

Hitherto I have fuppofed, that the conftant quantities of the equation are fuch, as may make room for the aforefaid fubftitutions. If they fhould not be

such, as, for example, if the equation were $aa - bx = yy$, we must make $aa = bc$, and then we shall have $bc - bx = yy$, and the substitution to be made would be that of $c - x$ equal to a new unknown quantity. Thus, if it were $\frac{abb}{m} + cx = yy$, we must make $bb = cf$, whence the equation $\frac{acf}{m} + cx = yy$. And then we must put $\frac{af}{m} + x$ equal to some new unknown quantity.

If it were $\frac{aax - bbx + m^3}{a + b} = yy$, we might make $aa - bb = cc$, and $m^3 = ccf$, and then it would be $\frac{ccx + ccf}{a + b} = yy$. And the like of others.

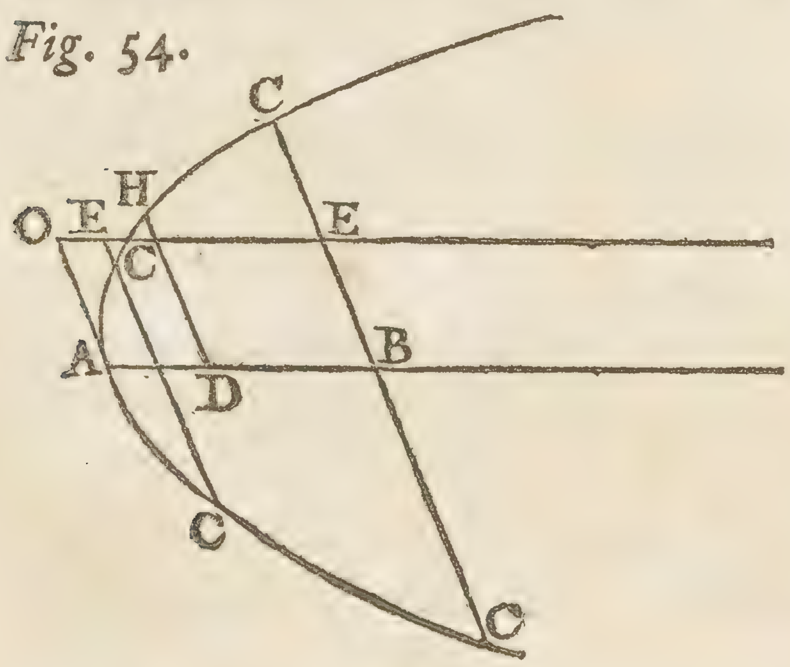
Loci of the second species constructed.

128. To reduce and construct equations of the second species; let all the terms which contain the same unknown quantity be put in order on one side of the sign of equality, and on the other side all the other terms in order likewise; and in the first member of the equation let the square of the unknown quantity be positive, and free from co-efficients and fractions. To the same first member, (and to the second also, to preserve the equality,) must be added the square of half the co-efficient of the second term, if it be necessary, so as the first member may be a square. Then put the root of that square equal to a new unknown quantity; which operation must be performed in the second member also, if it require it. This will give us an equation reduced to the simplest terms, or to an equation of the first species.

EXAMPLE III.

Let the equation be $xx + 2ax = ay + by$. Add the square aa on each side, and it will be $xx + 2ax + aa = aa + ay + by$. And now, making $x + a = z$, we shall have $zz = aa + ay + by$, which is now reduced to the first species. Then, making $a + b = c$, and $aa = cf$, it will be $cf + cy = zz$; and putting $f + y = p$, it will be $zz = cp$, an equation to the *Apollonian* parabola.

Fig. 54.



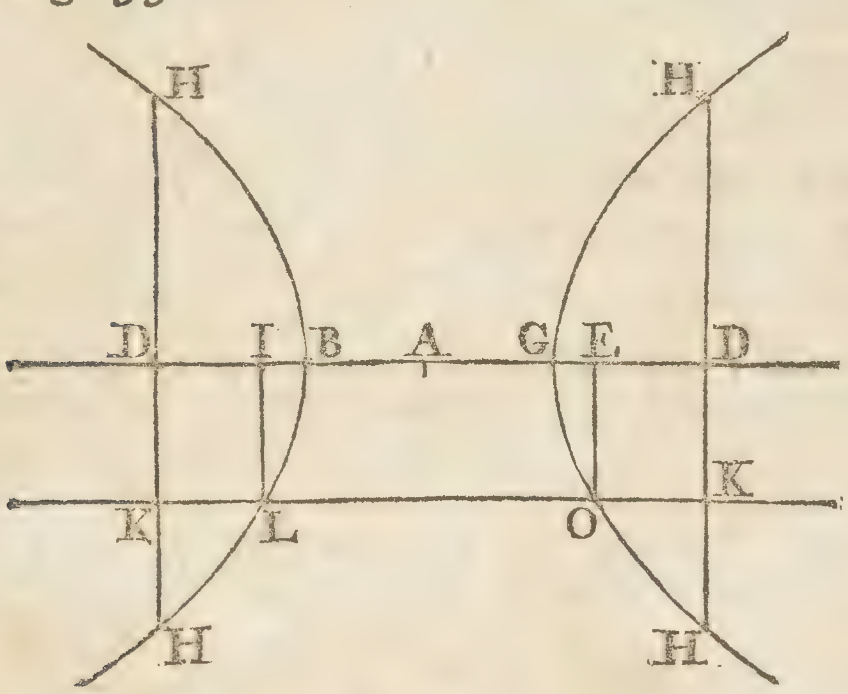
With parameter $c = a + b$, on the diameter AB, and with the co-ordinates in a given angle, let the parabola CAC be described. Then, taking any abscis $AB = p$, and BC shall be z, either positive or negative. And, because $y = p - f = p - \frac{aa}{a+b}$, taking $AD = \frac{aa}{a+b}$, it will be $DB = y$. And, because of the substitution $x + a = z$, from

from the point D draw DH $= a$ parallel to BC, which will be terminated by the parabola in H, (as will eaſily be ſeen by ſubſtituting, inſtead of p in the reduced equation $zz = cp$, the value of AD $= \frac{aa}{a+b} = \frac{aa}{c}$; for it will become $zz = aa$, and therefore DH $= z = a$,) and drawing through the point H the line OE parallel to the diameter, it will be HE $=$ DB $= p - \frac{aa}{a+b} = y$, and conſequently EC $= z - a = x$ poſitive, and negative alſo when the abſciſſes are poſitive. And to the negative abſciſſes, that is, taking them from H towards O, both the negative ordinates will correſpond.

EXAMPLE IV.

Let the equation be $xx + 2bx = yy - ay$. Let there be added the ſquare of half the co-efficient of the ſecond term, that is bb; then it will be $xx + 2bx + bb = yy - ay + bb$. And making $x + b = z$, we ſhall have $zz = yy - ay + bb$, that is, $zz - bb = yy - ay$. And adding the ſquare of $\frac{1}{2}a$, it will be $zz - bb + \frac{1}{4}aa = yy - ay + \frac{1}{4}aa$. Then make $y - \frac{1}{2}a = p$, and it will be $zz - bb + \frac{1}{4}aa = pp$. And ſuppoſing bb greater than $\frac{1}{4}aa$, and making $bb - \frac{1}{4}aa = mm$, it will be $zz - mm = pp$, an equilateral hyperbola with the ſemidiameters $= m$, and taking the abſciſſes from the centre.

Fig. 55.



In the indefinite line BD I take BG $= 2m = 2\sqrt{bb - \frac{1}{4}aa}$, and divide it equally in A. With centre A, the tranſverſe diameter $=$ 2AG, equal to the conjugate, and with the co-ordinates in a given angle, deſcribe the two oppoſite and equilateral hyperbolas. Taking any abſciſs poſitive and negative AD $= z$, the correſponding ordinates DH will be p, poſitive and negative. And becauſe, by the ſubſtitution, it is $x = z - b$, taking AE $= b$, it will be ED $= x$. But, by the other ſubſtitution, it being $y = p + \frac{1}{2}a$, from the point E drawing EO $= \frac{1}{2}a$, parallel to the ordinate, which will terminate at the curve in the point O; and through that point O draw the indefinite line KK parallel to the diameter BG, it will be KH $= p + \frac{1}{2}a = q$. Therefore the point O will be the origin of the abſciſs x on the right line KK, to which, taken poſitively, will correſpond the two ordinates y, one poſitive and the other negative. And taking it negative, but not greater than EG, two

positive ordinates will correspond to it; but taking it negative and greater than EG, but less than EB, the ordinates y will be imaginary; and taking it negative greater than EB, and less than EI, making BI = GE, the two ordinates will be positive; and lastly, one of the ordinates will be positive, and the other negative, when the abscisses, being negative, shall be greater than EI.

Here it should be observed, that the root of the square $yy - ay + \frac{1}{4}aa$ is not only $y - \frac{1}{2}a$, but also $\frac{1}{2}a - y$, and therefore the substitutions should be two, that is, both $y - \frac{1}{2}a = p$, and $\frac{1}{2}a - y = p$. Yet, notwithstanding, in the present example, and in others that follow, I only make use of the first. For, considering, in these constructions, the new unknown quantity p is to be understood both as positive and negative, herein will be comprehended those determinations also, which the other substitution would supply, and which therefore would be superfluous here.

If the quantity bb, which I have supposed greater than $\frac{1}{4}aa$, should, on the contrary, be less, the *locus* would be to the same hyperbolas, only by changing the places of the co-ordinates and of the constant quantities. That is, the final equation would be $zz = pp - mm$, the construction of which is here omitted, because it is not different from the foregoing, only that the semidiameters here are each equal to $\sqrt{\frac{1}{4}aa - bb} = m$. Now, if it were $bb = \frac{1}{4}aa$, the *locus* would degenerate into a right line, as is plain.

Loci of the third species constructed.

129. To distinguish and construct equations of the third species, it is necessary that, putting the square of one of the unknown quantities made positive, and free from fractions and co-efficients, together with the rectangle of the same, on one side of the mark of equality, and on the other side all the remaining terms; adding to the first member (and consequently to the second also) such a fraction of the other unknown quantity, that the first member may be a square; then putting it's root equal to a new unknown quantity, and making the substitution; by means of which an equation may be had, reduced to a more simple expression, or to one of the two species before-mentioned.

Thus, in this equation, for example, $zz - \frac{2bzy}{a} = ay$, adding $\frac{bbyy}{aa}$ to both members, the first member will be a square, the root of which is $z - \frac{by}{a}$, which is to be put equal to a new unknown quantity p; and, making the substitution, the equation will be $pp = \frac{bbyy}{aa} + ay$, which is now reduced to the second species.

130. But

130. But it may be obferved, that fometimes the new unknown quantity to be introduced fhould be affected by fome conftant co-efficient, otherwife the conftructions would be much incumbered. For example, in the equation $xx \pm \frac{2bxy}{a} + \frac{bbyy}{aa} = \pm fy \pm hx$, the firft member of which, without any addition, is already a fquare, whofe root is $x \pm \frac{by}{a}$; if the term hx were not there, or being there, if we would eliminate x out of the equation, we might do it, by putting, inftead of x, it's value obtained by the fubftitution, fo that it may be expreffed by the new unknown quantity, and by y with conftant quantities; therefore the fubftitution of $x \pm \frac{by}{a} = z$ fhould be made.

Complicate *loci* of any fpecies reduced to fimple by fubftitutions; with examples.

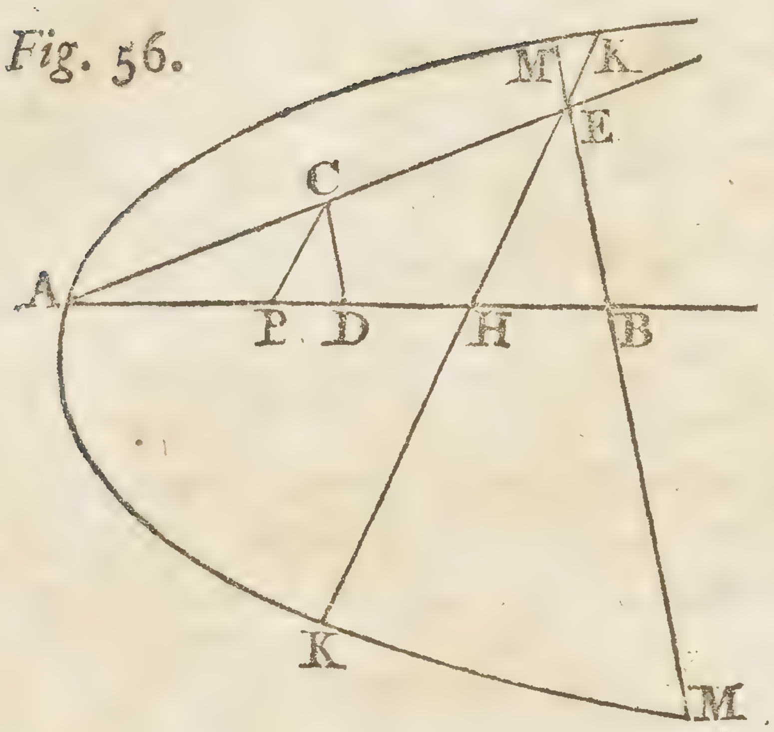
But if the term fy were not there, or being there, if we would eliminate y, we muft make a fubftitution of $x \pm \frac{by}{a} = \frac{bz}{a}$. And thus, refpectively, if the equation were $yy \pm \frac{2bxy}{a} + \frac{bbxx}{aa} = \pm fy \pm hx$, the term fy not being there, or elfe to be eliminated, a fubftitution muft be made of $y \pm \frac{bx}{a} = z$; or the term hx not being there, or being to be eliminated, a fubftitution of $y \pm \frac{bx}{a} = \frac{bz}{a}$ is to be made.

In general, the rectangle of conftant quantities into that unknown quantity by which the equation is ordered, not being in the equation; or being there, if we would eliminate that unknown quantity, we muft put the root of the firft member equal to a new unknown quantity. But if the rectangle of conftant quantities into the other unknown quantity, by which the equation is not ordered, be not in the equation, or if, being there, we would eliminate that unknown quantity, we muft put the root of the firft member equal to a new unknown quantity, multiplied into half the conftant co-efficient of the fecond term of the firft member.

EXAMPLE V.

Let the equation be $yy + \frac{2bxy}{a} + \frac{bbxx}{aa} = cx$. Make $y + \frac{bx}{a} = z$, and the equation will be $zz = cx$, which is to the *Apollonian* parabola. If the angle of the co-ordinates x, y, of the propofed equation be not given, but left at pleafure, the

Fig. 56.



the conſtruction of the *locus* would be manifeſt. For, on the indefinite right line AB deſcribing the iſoſceles triangle ACD, with the baſe CD $= b$, and the ſides AC $=$ AD $= a$; and on the diameter AB, with a parameter $= c$, and with ordinates parallel to DC deſcribing the parabola of the reduced equation $zz = cx$; taking any abſciſs at pleaſure AB $= x$, it would be BM $= z$. But, by the ſimilar triangles ADC, ABE, we ſhall have EB $= \frac{bx}{a}$, and, by the ſubſtitution, it is $y = z - \frac{bx}{a} =$ EM, and alſo AE $=$ AB $= x$. Therefore, upon the indefinite line AE taking any abſciſs AE $= x$, the correſponding ordinate EM, poſitive or negative, will be the y of the propoſed equation. But, becauſe the angle of the co-ordinates x and y is ſuppoſed to be given, the conſtruction aforegoing will not obtain, but we may proceed thus. On the indefinite line AB let a triangle ACP be deſcribed, having the angle ACP equal to the ſupplement of the given angle, which the co-ordinates of the propoſed equation ought to make; and let AC$=a$, CP $= b$. Produce AC indefinitely, and, taking any line AE $= x$, make KK parallel to PC, and it will be EH $= \frac{bx}{a}$. Whence, if HK $= z$, it will be EK $= y$; and then AE, EK, are the co-ordinates of the propoſed equation, and in the angle given. But HK cannot be yet the z of the reduced equation $cx = zz$, ſince the abſciſſes AH are not yet equal to the x's, nor yet the lines AE. Obſerve, therefore, that AH will be $= \frac{\text{AP} \times x}{a}$, that is, $= \frac{fx}{a}$, (making AP $= f$, becauſe, in the triangle ACP, having given the ſides AC, CP, and the angle ACP, the line AP will alſo be given ;) whence the curve thus deſcribed, calling AE $= x$, and HK $= z$, will give us the equation $\frac{cfx}{a} = zz$, which would be exactly our equation reduced, if, inſtead of the parameter c, we had deſcribed the curve with the parameter $\frac{ac}{f}$. Therefore, to conſtruct the propoſed *locus*, on the indefinite line AB deſcribe the triangle ACP, the ſides of which are AC$=a$, CP $= b$, and the angle ACP equal to the ſupplement of that angle which the co-ordinates of the propoſed equation ought to make. Then with diameter AB, parameter $= \frac{ac}{f}$, equal to the fourth proportional of AP, of AC, and of the parameter of the reduced equation, (which is general, whenever the *locus* is to

 the

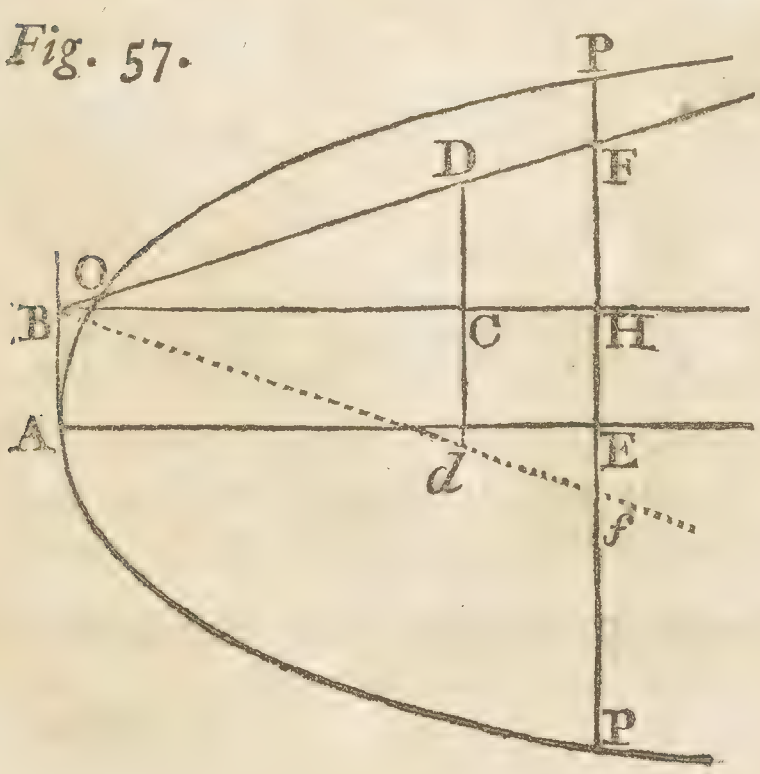
the parabola,) and with ordinates parallel to PC, the *Apollonian* parabola muſt be deſcribed. Then taking, on the indefinite line AE, any abſciſs AE $= x$, EK poſitive and negative will be $= y$, and the curve will be the *locus* of the equation propoſed. For it will be HKq equal to the rectangle of the parameter into AH, or $yy + \frac{2bxy}{a} + \frac{bbxx}{aa} = \frac{acfx}{af} = cx$.

The ſame artifice may be made uſe of in other equations, to the hyperbola and to the ellipſis, in regard to their diameters and parameters, with this difference only, that in theſe the tranſverſe diameter, or conjugate, according as this or that ought to be changed, (and it will always be that to which the triangle ACP belongs,) will be the fourth proportional of AC, AP, and the tranſverſe or conjugate diameter of the equation reduced. But as to the parameter, when the equation is given by that, the tranſverſe diameter being varied in the manner aforegoing, it will be the fourth proportional of AP, AC, and the parameter of the reduced equation. But if the triangle ACP do not belong to the tranſverſe diameter, but to the conjugate, (the equation being given by the parameter,) it will be the third proportional of the parameter of the reduced equation, and of AP; as will eaſily be known by the examples.

EXAMPLE VI.

Let the equation given be $yy + \frac{2bxy}{a} + \frac{bbxx}{aa} = bx - cc - 2cy$. Making a ſubſtitution of $y + \frac{bx}{a} = z$, it will be $zz = bx - cc - 2cz + \frac{2bcx}{a}$, that is $zz + 2cz + cc = bx + \frac{2bcx}{a}$. And making again another ſubſtitution of $z + c = q$, it will be finally $qq = \frac{ab + 2bc}{a}x$, an equation to the *Apollonian* parabola. Now, to conſtruct it relatively to our co-ordinates x, y; on the indefinite right line BH let the triangle BCD be conſtructed with it's ſides BD $= a$, DC $= b$, and with an angle BDC equal to the ſupplement of that angle, which ought to be made by the co-ordinates x, y, of the equation propoſed. Then let BD, BC, be produced indefinitely, and from the point B draw BA parallel to DC, and equal to c. Then from vertex A to the diameter AE parallel to BC, and with the ordinates EP parallel to CD, let the parabola

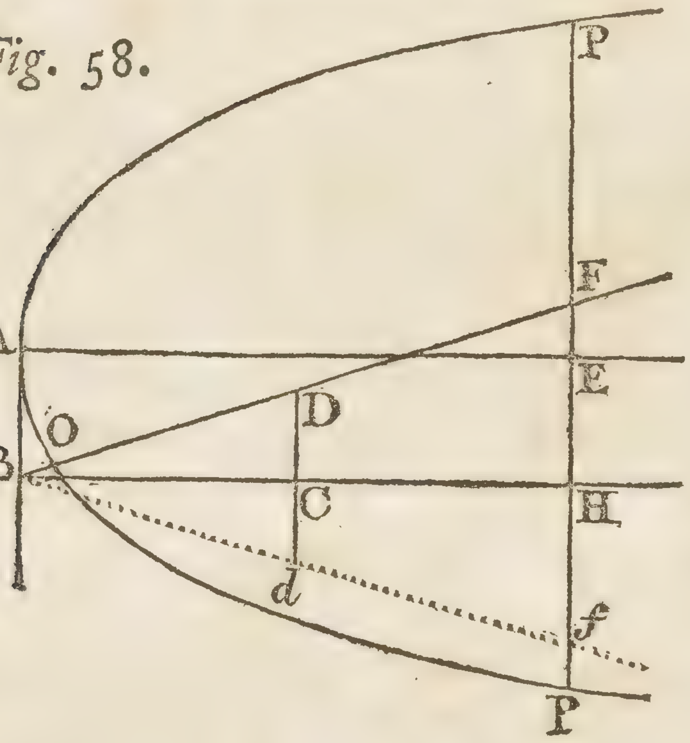
Fig. 57.



parabola PAP be defcribed, with the parameter $= \frac{ab + 2bc}{f}$, (meaning by f the known line BC,) and on the indefinite line BF taking any abfcifs BF $= x$, it will be BH $=$ AE $= \frac{fx}{a}$, and EP $= q$, and therefore HP $= q - c = z$, and FH $= \frac{bx}{a}$. Then FP $= z - \frac{bx}{a} = y$, pofitive and negative when x is greater than BO; and both ordinates negative, when it is x lefs than BO.

In the equation propofed, if the rectangle $2cy$ fhall be affected by the affirmative fign, then the fecond fubftitution fhould be $z - c = q$, and the parameter of the

Fig. 58.

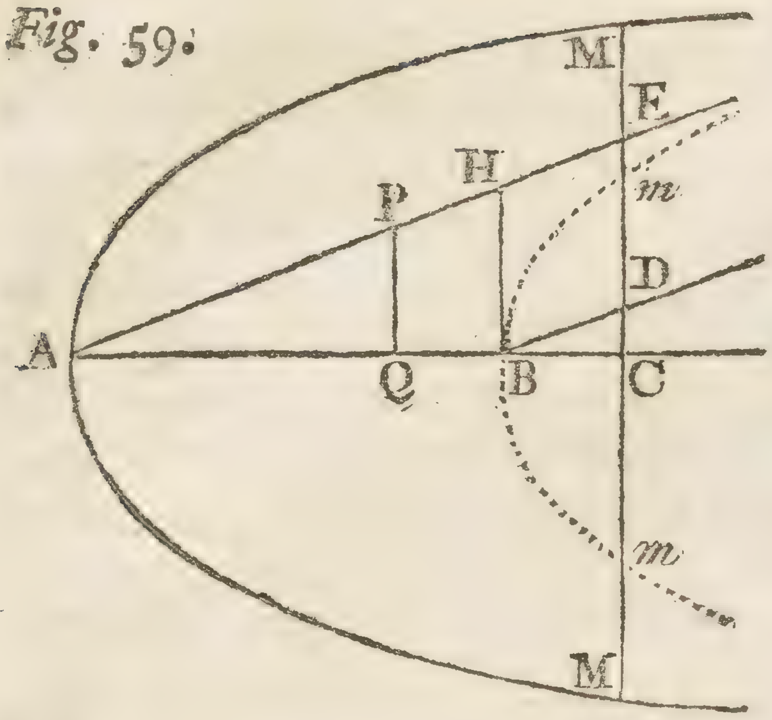


parabola equal to $\frac{ab - 2bc}{a}$. Then doing the fame things as before, inftead of drawing BH above the diameter AE, it fhould be drawn below it, and the triangle BDC fhould be made above it, as is fhown by Fig. 58. Moreover, if the term $\frac{2bxy}{a}$ be negative, the firft fubftitution fhould be $y - \frac{bx}{a} = z$, and thence $y = z + \frac{bx}{a}$. Therefore, in this fuppofition, as well in regard to Fig. 57 as Fig. 58, the triangle BDC fhould be conftructed below BH, fuppofe as BdC. Wherefore, taking any line Bf $= x$ above Bd produced, it will be fP $= y$; obferving that, in this cafe, the angle BdC fhould not be made equal to the fupplement, but to the angle itfelf, which is to be made by the co-ordinates of the equation.

EXAMPLE VII.

Let the equation be $xx + \frac{2bxy}{a} + \frac{bbyy}{aa} = cx + cb$. Making the fubftitution of $x + \frac{by}{a} = \frac{bz}{a}$, it will be $\frac{bbzz}{aa} = cx + cb$; and making $x + b = p$, it will be $zz = \frac{aacp}{bb}$, an equation to the *Apollonian* parabola. On the indefinite

Fig. 59.



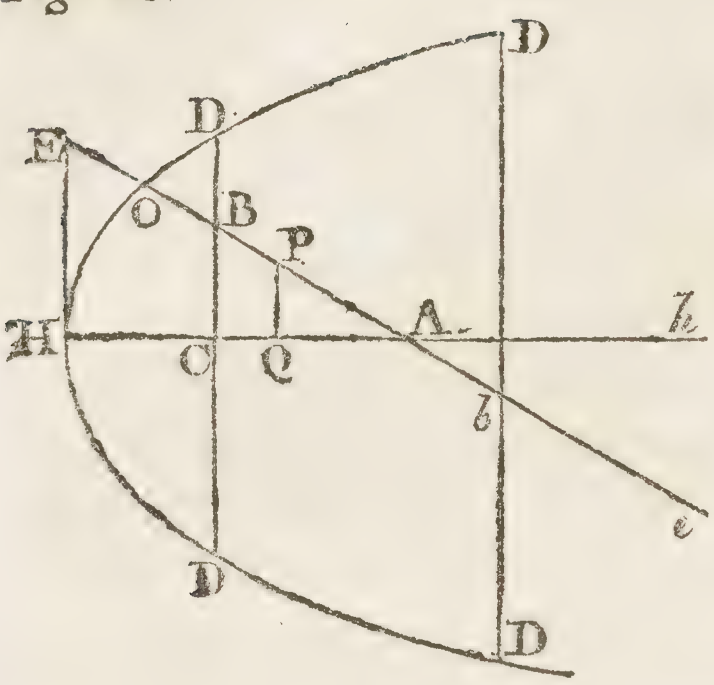
finite line AC describe the triangle APQ with the sides AP $= b$, PQ $= a$, and the angle APQ equal to the supplement of the angle which should be made by the co-ordinates of the proposed equation; and call the known line AQ $= f$, as usual. Let AP, AQ be produced indefinitely, take AH $= h$, and draw the line HB parallel to PQ. From the point B let the indefinite line BD be drawn parallel to AP; and with vertex A, to the diameter AC, with the parameter $= \frac{aac}{bf}$, and with the ordinate CM parallel to PQ, let the parabola MAM be described. Taking any line AE $= p$, it will be CM $= z$; then HE or BD $= x$, and DC $= \frac{ax}{b}$, because of the similar triangles APQ, BDC. Then is DM $= z - \frac{ax}{b} = y$ positive and negative, and the lines BD, DM, are the co-ordinates of the proposed equation.

If the equation had been given $xx + \frac{2bxy}{a} + \frac{bbyy}{aa} = cx - ch$, making the same first substitution as in the foregoing equation, we should have $\frac{bbzz}{aa} = cx - ch$; and, putting $x - h = p$, it is $zz = \frac{aacp}{bb}$, which is the same as the first, nor is there any other difference, but only in the first case there is $x = p - h$, and here it is $x = p + h$. That is, in the present case the vertex of the parabola must be at B, and the origin of the abscis x must be in the point A, taken on the indefinite line AE.

EXAMPLE VIII.

Let the equation be $xx + \frac{2bxy}{a} + \frac{bbyy}{aa} = ch - cx$. Make the substitution of $x + \frac{by}{a} = \frac{bz}{a}$, and the equation will be $\frac{bbzz}{aa} = ch - cx$; and putting $h - x = p$, it will be $zz = \frac{aacp}{bb}$, an equation to the parabola.

Fig. 60.



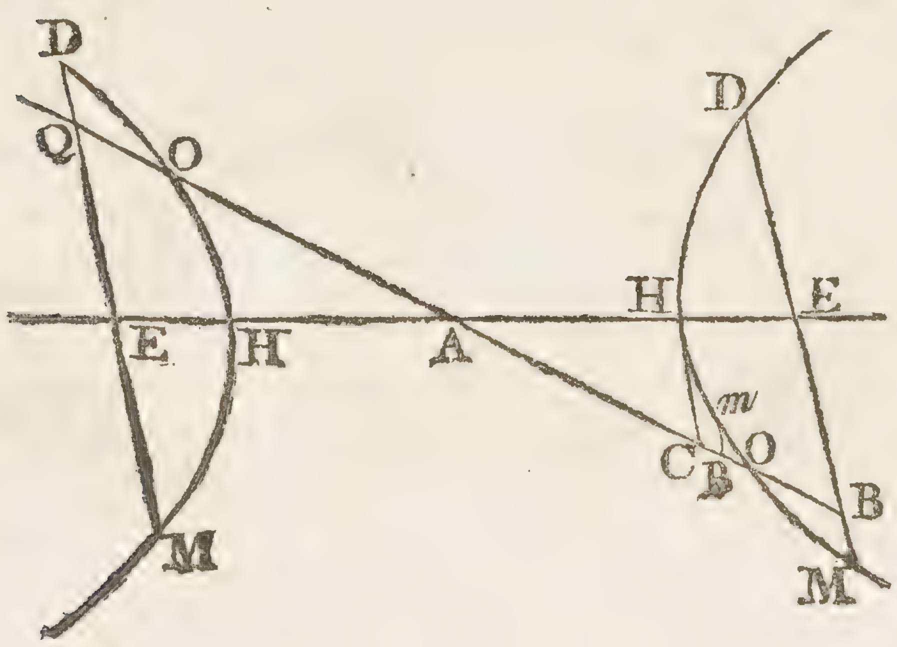
On the indefinite line AH let the triangle APQ be deſcribed towards H, with the ſides AP $= b$, PQ $= a$, and the angle APQ equal to the ſupplement of the angle which the co-ordinates of the propoſed equation ought to contain. Make the known line AQ $= f$. Produce AP, and take AE $= b$, and draw EH parallel to PQ. With vertex H, on the diameter HA, with the ordinates CD parallel to PQ, and with the parameter $= \frac{aac}{bf}$, let there be deſcribed the Apollonian parabola. Taking any line EB $= p$, it will be AB $= b - p = x$, BC $= \frac{ax}{b}$, CD $= z$. Then is BD $= z - \frac{ax}{b} = y$ poſitive and negative, taking x between the points A and O; and both the ordinates y negative, taking x beyond the point O. The right line AE being produced indefinitely on the oppoſite ſide to the point E, and taking any line Eb $= p$ poſitive and greater than AE, it will be Ab $= b - p = x$, a negative quantity; whence in this caſe the negative x's will be from A towards e, and the poſitive from A towards E; and to the ſame negative x will correſpond two ordinates bD, bD, equal to y, one poſitive and the other negative.

If in theſe two laſt examples, as in the others which will follow, the rectangle of the two ordinates be affected by the ſign $-$, it is done upon the ſame conſideration as is mentioned at the end of the 6th Example; which it may ſuffice to have mentioned once for all.

EXAMPLE IX.

Let the equation be $yy - \frac{2bxy}{a} + \frac{bbxx}{aa} = xx - aa$. Make the ſubſtitution of $y - \frac{bx}{a} = z$, and the equation will be $zz = xx - aa$, which is to the hyperbola. On the indefinite line EE deſcribe the triangle ACH, and make AC $= a$, CH $= b$, and the angle ACH equal to the given angle of the co-ordinates of the equation propoſed. Let AC be produced indefinitely both ways from the point A. With centre A, and tranſverſe ſemidiameter AH $= f$, with the conjugate $= a$, let there be deſcribed the oppoſite hyperbolas with the ordinates

Fig. 61.

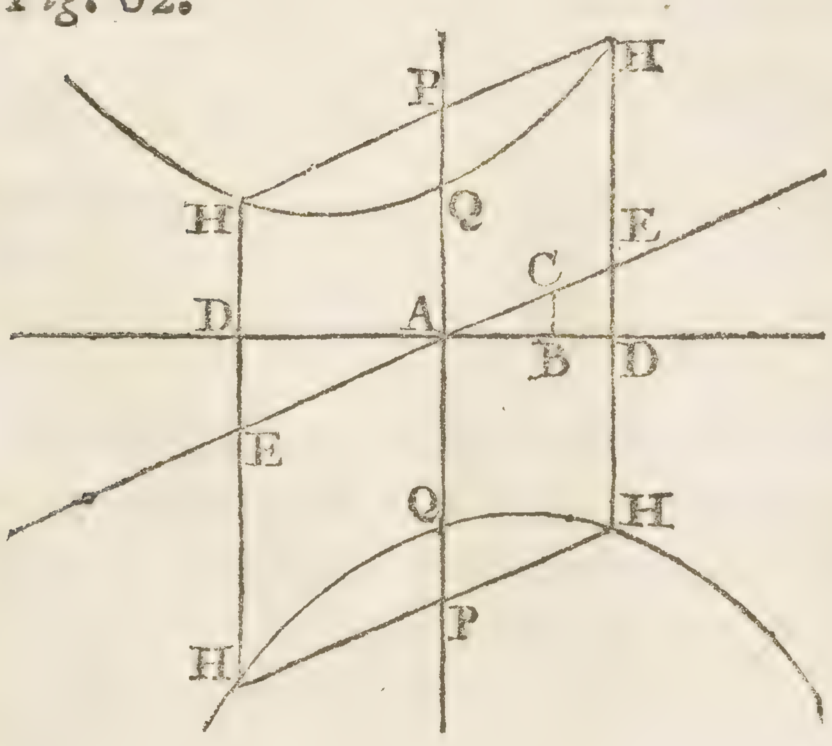


nates parallel to CH. Taking any line AB $= x$ positive, it will be BE $= \frac{bx}{a}$. But ED $= z$: Then is BD $= z + \frac{bx}{a} = y$ positive. And taking in the hyperbola the ordinate z negative, that is $=$ EM, then will y be equal to the difference between EB and EM, that is, equal to BM; and therefore negative when x is greater than AO. Then to any positive abscifs greater than AO will correspond two ordinates, one positive and the other negative; and both the ordinates will be positive when x is less than AO. But when x is taken negative, that is on the side of the point Q, then it must be observed that QE will be negative; for the analogy will be, AC (a) . CH (b) :: AQ $(-x)$. QE $= -\frac{bx}{a}$. Therefore, if QE $= -\frac{bx}{a}$, taking z positive $=$ ED, it will be $z + \frac{bx}{a} =$ QD $= y$ positive; and taking z negative, it will be $-z - \frac{bx}{a} =$ QM $= y$ negative.

EXAMPLE X.

Let the equation be $yy - \frac{2bxy}{a} + \frac{gxx}{a} = hh$. Adding $\frac{bbxx}{aa}$, it will be $yy - \frac{2bxy}{a} + \frac{bbxx}{aa} = hh - \frac{gxx}{a} + \frac{bbxx}{aa}$; and making the substitution of $y - \frac{bx}{a} = z$, it will be $zz = \frac{bbxx}{aa} - \frac{gxx}{a} + hh$. And putting $bb - ag = mm$, it will be $zz = \frac{mmxx}{aa} + hh$, that is, $zz - hh = \frac{mmxx}{aa}$, an equation to the hyperbola.

Fig. 62.



On the indefinite right line DD let the triangle ABC be described, with the sides AB $= a$, BC $= b$, and the angle ABC equal to that which is to be contained by the co-ordinates of the proposed equation; and make the known line $= f$. Through the point A draw the indefinite line PP parallel to BC, and with centre A, transverse diameter QQ $= 2b$, conjugate $= \frac{2hf}{m}$ taken in the right line EE, at the vertices Q, Q, let there be described the two opposite hyperbolas HQH. Then taking

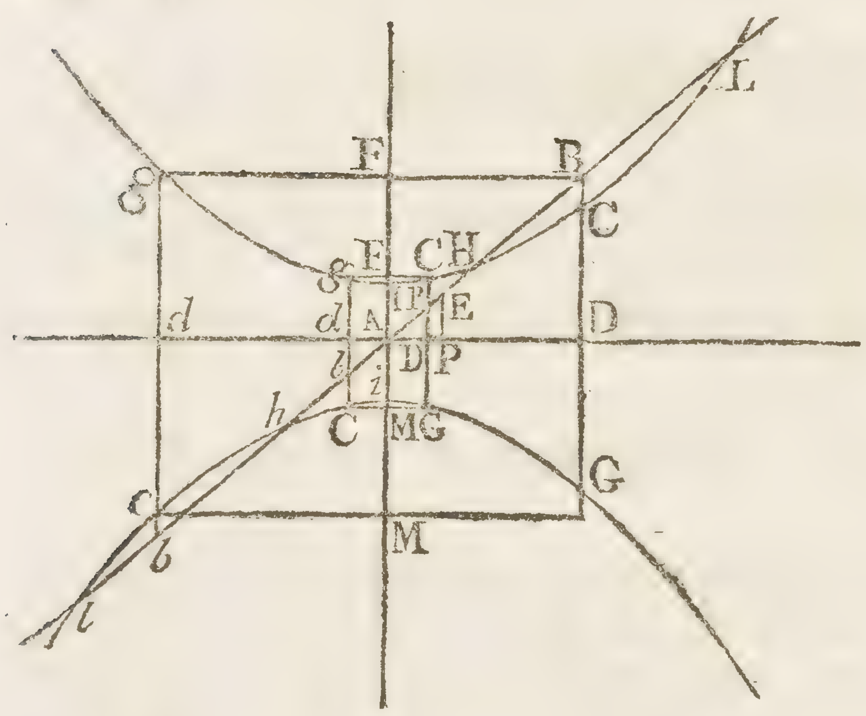
 any

any line AD $= x$, and drawing DH parallel to BC, it will be EH $= z =$ AP, and DE $= \frac{bx}{a}$. Then DH $= z + \frac{bx}{a} = y$, and the lines AD, DH, ſhall be the co-ordinates of the propoſed equation.

EXAMPLE XI.

Let the equation be $yy + \frac{2bxy}{a} + \frac{bbxx}{aa} = \frac{2bxx}{a} + bb$. Making the ſubſtitution of $y + \frac{bx}{a} = z$, the equation will be $zz = \frac{2bxx}{a} + bb$, that is, $zz - bb = \frac{2bxx}{a}$, which is to the hyperbola.

Fig. 63.



On the indefinite line AD let the triangle AEP be deſcribed, and make AE $= a$, EP $= b$, and the angle AEP the ſupplement of the angle, which is to be contained by the co-ordinates of the propoſed equation. The right line AE being produced indefinitely both ways, and calling, as uſual, the known line AP $= f$; with centre A, tranſverſe ſemidiameter AI $= b$ parallel to PE, and with parameter $= \frac{ff}{a}$, deſcribe the oppoſite hyperbolas IC, ic; then taking any line AB $= x$, it will be BD $= \frac{bx}{a}$, and CD $=$ FA $= z$. Then BC $= z - \frac{bx}{a} = y$. Taking z negative $=$ DG, it will be BG $= -z + \frac{bx}{a} = -y$, and therefore to the ſame poſitive x will belong two ordinates y, one poſitive, the other negative, taking x between the points A, H. Then taking x between the points H, L, both the ordinates y will be negative; and again, one poſitive, the other negative, taking x greater than AL.

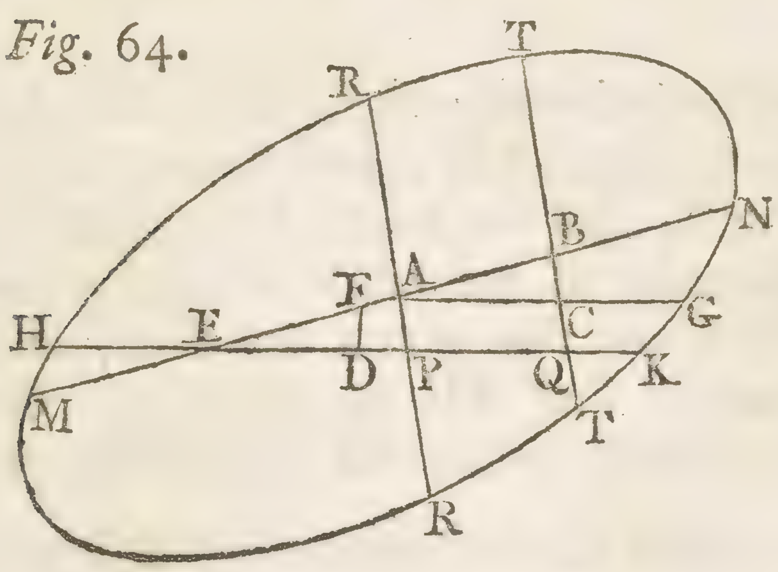
Then

Then taking $Ab = -x$, it will be $(bd) = -\frac{bx}{a}$, and as it is $(dg) = z$, it will be $(bg) = z - \frac{bx}{a} = y$; and taking z negative $= (dc)$, it will be $(bc) = -z + \frac{bx}{a} = -y$. Therefore to the same $Ab = x$ negative will correspond two ordinates y, one of which is positive, the other negative, taking x less than Ah; both the ordinates will be positive between the points h and l; and again, one ordinate will be positive, and the other negative, taking x greater than Al. And therefore the hyperbolas thus described will be the *locus* of the proposed equation.

EXAMPLE XII.

Let the equation be $yy - \frac{2bxy}{a} + \frac{bbxx}{aa} = cc - xx + 2bx - bb$. Making the substitution of $y - \frac{bx}{a} = z$, it will be $zz = cc - xx + 2bx - bb$. And making another substitution of $x - b = p$, it will be finally $zz = cc - pp$, which is an equation to an ellipsis, and not to a circle, though it may have the appearance of such. The reason of which is, because the co-ordinates p, z, do not form a right angle, yet however are in an angle to each other, one of them being AC, the other BT, as may be seen in the following construction. On the indefinite line EB let a triangle EDF be described, with the sides ED $= a$, DF $= b$, and the angle EDF equal to the angle which is made by the co-ordinates of the proposed equation; and making the known line EF $= f$. Produce indefinitely the lines ED, EF, and taking EP $= b$, draw the indefinite line PA parallel to DF, and from the point A the line AG parallel to EP.

Fig. 64.



With centre A, transverse diameter MN $= \frac{2cf}{a}$, with conjugate diameter RR, equal to $2c$ and parallel to DF, let the ellipsis MRNR be described; then taking any line AC $= p$, it will be EQ $= x$, and therefore BQ $= \frac{bx}{a}$. But BT $= z$; then QT $= z + \frac{bx}{a} = y$; then will EQ, QT, be the co-ordinates of the *locus* required.

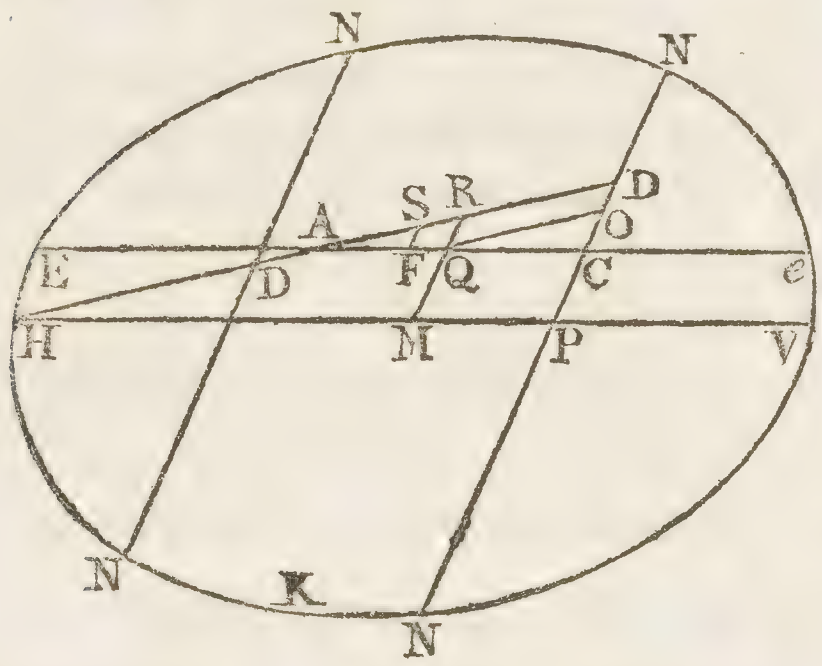
EX-

EXAMPLE XIII.

Let the equation be $yy + \frac{bxy}{a} + xx + cy + lx - ag = 0$. Adding on both ſides the ſquare $\frac{bbxx}{4aa}$, it will be $yy + \frac{bxy}{a} + \frac{bbxx}{4aa} = \frac{bbxx}{4aa} - xx - lx - cy + ag$. And making the ſubſtitution of $y + \frac{bx}{2a} = z$, it will be $zz = \frac{bbxx - 4aaxx}{4aa} + \frac{bcx - 2alx}{2a} - cz + ag$.

Let $4aa$ be greater than bb, and make $\frac{bb - 4aa}{4aa} = -\frac{m}{n}$, and $\frac{bc - 2al}{2a} = h$; then adding $\frac{1}{4}cc$ on each ſide, it will be $zz + cz + \frac{1}{4}cc = -\frac{mxx}{n} + hx + ag + \frac{1}{4}cc$. And making the ſubſtitution of $z + \frac{1}{2}c = p$, it will be $pp = -\frac{mxx}{n} + hx + ag + \frac{1}{4}cc$. That is, $-\frac{npp}{m} + \overline{\frac{1}{4}cc + ag} \times \frac{n}{m} = xx - \frac{nhx}{m}$; and laſtly, adding $\frac{nnhh}{4mm}$ to both ſides, and making the ſubſtitution of $x - \frac{nh}{2m} = q$, and of $\overline{\frac{1}{4}cc + ag} \times \frac{n}{m} + \frac{nnhh}{4mm} = ee$, we ſhall have $\frac{npp}{m} = ee - qq$, which is an equation to the ellipſis.

Fig. 65.



Upon the indefinite right line AC deſcribe the triangle ASF, and make AS $= 2a$, SF $= b$, and the angle ASF equal to the ſupplement of the angle made by the co-ordinates of the given equation, and let the known line AF be called f. On AS indefinitely produced take AR $= \frac{hn}{2m}$, and draw the indefinite line RQ parallel to FS, and from the point Q draw the indefinite line QO parallel to AS, and make QM $= \frac{1}{2}c$.

Then through the point M draw HV parallel to AQ, and with centre M, tranſverſe diameter HV $= \frac{ef}{a}$, and parameter $= \frac{4aem}{fn}$, let the ellipſis HNVK be

be deſcribed. And taking any line RD $= q$, it will be PN $= p$, and therefore AD $= x$, DC $= \frac{bx}{2a}$, CN $= z$; then DN $= z - \frac{bx}{2a} = y$.

Here it is to be obſerved, that if the angle of the co-ordinates ſhould be ſuch, as that the angle AFS becomes a right angle, and conſequently the angle MPN is ſo too; then it would be $4aa - bb = ff$, whence $\frac{m}{n} = \frac{4aa - bb}{4aa} = \frac{ff}{4aa}$, and therefore the parameter would be $\frac{4aem}{fn} = \frac{ef}{a}$, that is, equal to the tranſverſe diameter. Then the angle MPN being alſo right, the ellipſis would degenerate into a circle with the diameter $= \frac{ef}{a}$.

131. As to equations of the hyperbola between the aſymptotes, which may be required to be conſtructed, they may all be underſtood to be comprehended in the four examples following. General conſtruction of the *loci* to the hyperbola between it's aſymptotes; with examples.

$$(1.)\quad \frac{gxx}{b} + xy = ab \pm mx \pm ny.$$

$$(2.)\quad -\frac{gxx}{b} + xy = ab \pm mx \pm ny.$$

$$(3.)\quad \frac{gxx}{b} - xy = ab \pm mx \pm ny.$$

$$(4.)\quad -\frac{gxx}{b} - xy = ab \pm mx \pm ny.$$

EXAMPLE XIV.

Firſt, let the equation be $\frac{gxx}{b} + xy = ab + mx + ny$, in which I take all the terms poſitive of the *homogeneum comparationis*. Making a ſubſtitution of $\frac{gx}{b} + y = z$, we ſhall have $zx = mx + nz - \frac{ngx}{b} + ab$; and, making another ſubſtitution of $z - m + \frac{ng}{b} = p$, it will be $px = np + mn + ab - \frac{nng}{b}$. Again, make a third ſubſtitution of $x - n = q$, and, finally, it will be $pq = ab + nm - \frac{nng}{b}$. Suppoſing now that $ab + mn - \frac{nng}{b}$ is a poſitive quantity;

Fig. 66.

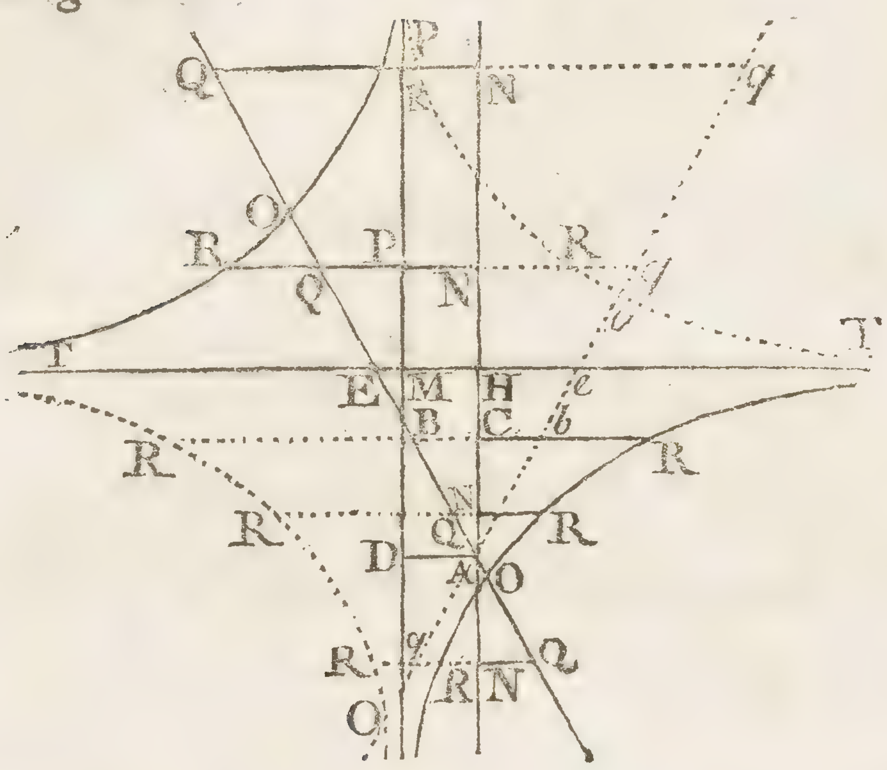
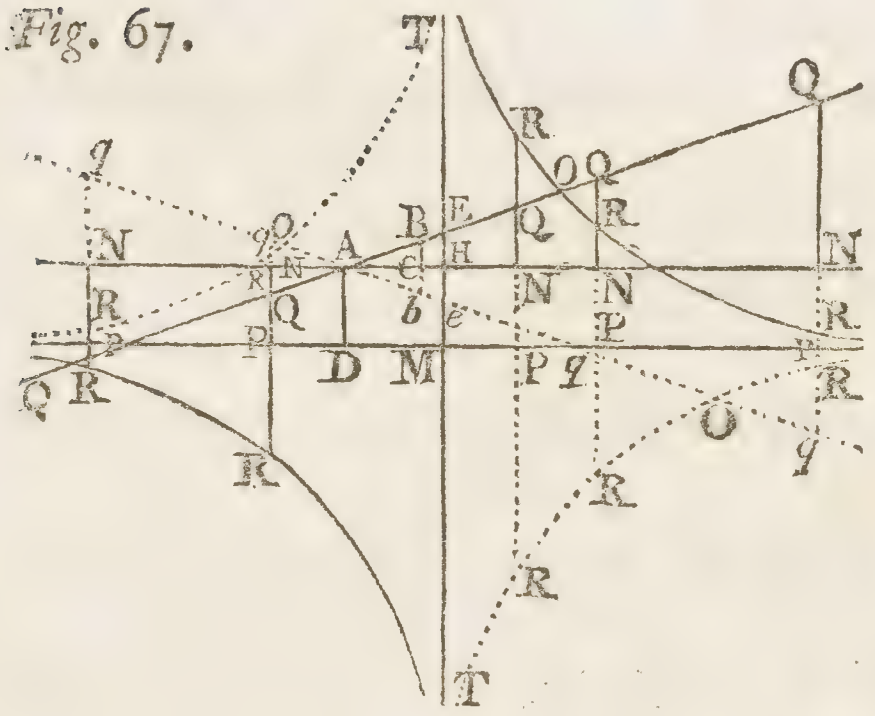


Fig. 67.



on the indefinite line NN, at the point A taken at pleafure, defcribe the triangle ABC, the fides of which are AB $= b$, BC $= g$, and the angle ABC equal to the fupplement of the angle which the co-ordinates of the equation propofed ought to make, and make the known line AC $= f$. At the point A raife AD parallel to BC, and equal to $m - \frac{ng}{b}$, as in Fig. 66, when $m - \frac{ng}{b}$ is a pofitive quantity; and let fall AD, as in Fig. 67, when $m - \frac{ng}{b}$ is a negative quantity, becaufe of the fubftitution made of $z - m + \frac{ng}{b} = p$. Through D draw the indefinite line PP parallel to AC, and on AB produced take AE $= n$, and through E draw TT parallel to BC. Between the afymptotes PP, TT, defcribe the two oppofite hyperbolas RR of the conftant rectangle $= \overline{ab + mn - \frac{nng}{b}} \times \frac{f}{b}$, that is, a fourth proportional to AB, AC, and the conftant rectangle of the equation reduced. Taking any line EQ $= q$, it will be PM $= \frac{fq}{b}$, and PQ $= p$, and therefore AQ $= q + n = x$. But PN $=$ AD $= m - \frac{ng}{b}$, therefore NR $= p + m - \frac{ng}{b} = z$; and becaufe QN $= \frac{gx}{b}$, it will be, laftly, QR $= z - \frac{gx}{b} = y$, and the two lines AQ, QR, will be the co-ordinates of the propofed equation. Taking x pofitive, when it is lefs than AE, y will be negative: when it is greater than AE, and lefs than AO, y will be pofitive, and when it is greater than AO, y will be negative. Taking x negative, then it will be QN $= -\frac{gx}{b}$, a negative quantity; then $y = z - \frac{gx}{b}$ will be $=$ NR $+$ NQ; and therefore, when x negative is lefs than AO, y will be negative; and when it is greater than AO, y will be pofitive.

But

But if the ſecond term of the *homogeneum comparationis* ſhould be negative, that is, if the equation were $\frac{gxx}{b} + xy = ab - mx + ny$; then the ſecond ſubſtitution would be $z = p - m - \frac{ng}{b}$, and the equation reduced $pq = ab - mn - \frac{nng}{b}$. Suppoſing then that $ab - mn - \frac{nng}{b}$ were a poſitive quantity, deſcribe, as in Fig. 67, the hyperbolas RR, but with the conſtant rectangle $\overline{ab - mn - \frac{nng}{b}} \times \frac{f}{b}$, and taking $AD = m + \frac{ng}{b}$, this would be in the ſame manner the *locus* of the propoſed equation.

Fig. 68.

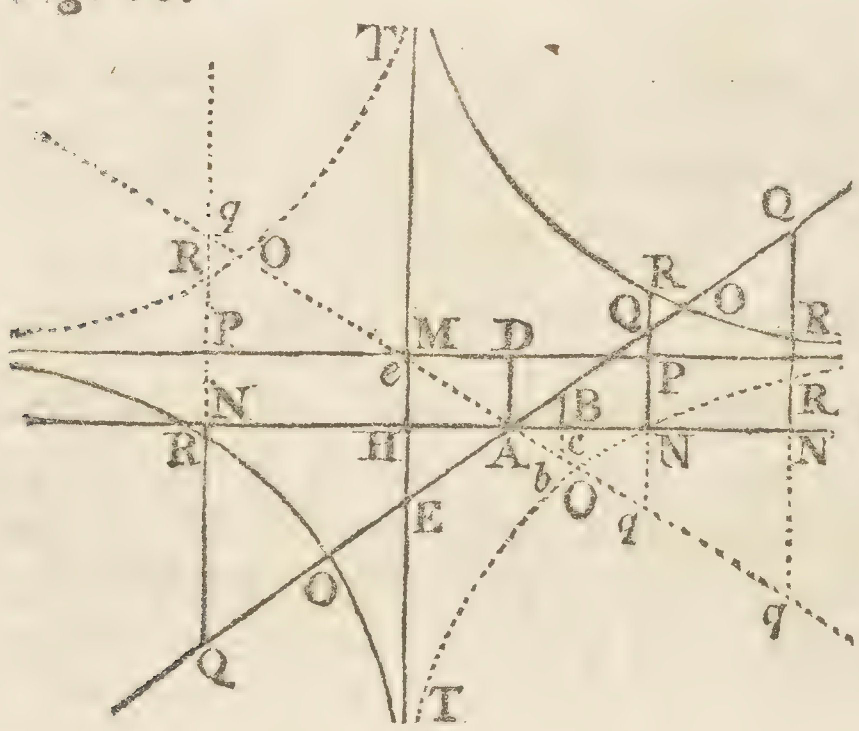
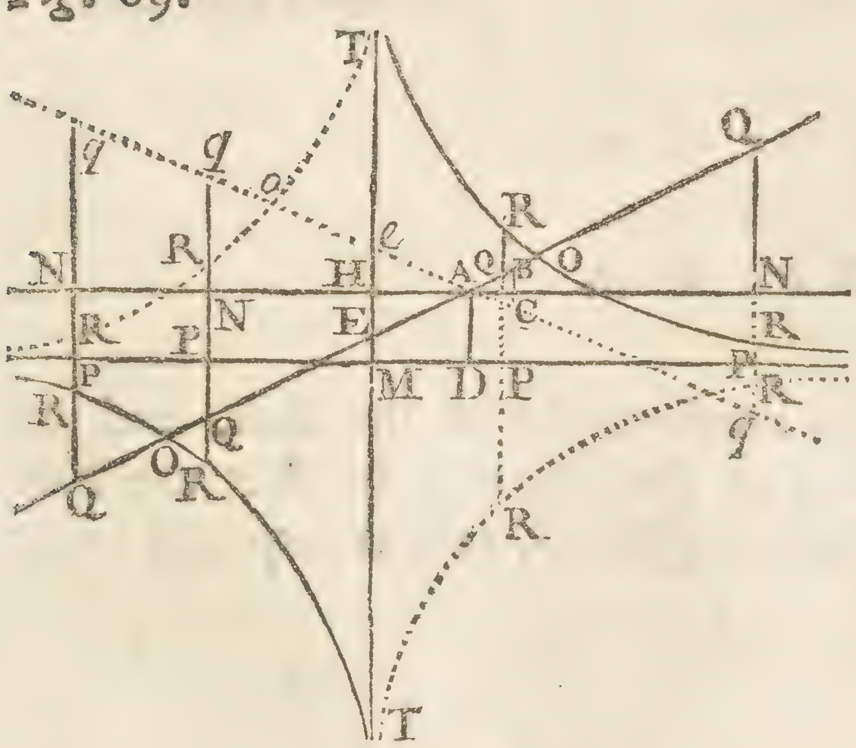


Fig. 69.



If the equation propoſed had the laſt term affected by the negative ſign, that is, if it were $\frac{gxx}{b} + xy = ab \pm mx - ny$, the third ſubſtitution to be made would be $x + n = q$, whereas before it was $x - n = q$, and therefore the poſition of the point A, the origin of x, would be changed. Then, in Fig. 68, if the value of AD be poſitive, and in Fig. 69, if it be negative, the ſide BA of the uſual triangle being produced to E, ſo that $AE = n$; between the aſymptotes TT, PP, let the hyperbolas be deſcribed of the conſtant rectangle belonging to them, that is, when in the equation the term mx is affected by the poſitive ſign, then the conſtant rectangle $= \overline{ab - mn - \frac{nng}{b}} \times \frac{f}{b}$, and when, on the contrary, it is affected by the negative ſign, the conſtant rectangle will be $= \overline{ab + mn - \frac{nng}{b}} \times \frac{f}{b}$; and taking, in the firſt caſe, $AD = m + \frac{ng}{b}$, and in the ſecond, $AD = \frac{ng}{b} - m$, the *locus* of the propoſed equation will be after the ſame manner.

Hitherto I have ſuppoſed, that the conſtant rectangle of the reduced equation is a poſitive quantity; but when it happens to be negative, the conſtruction

 would

would not be different, only obferve to defcribe the hyperbolas in the other two angles, relatively to the conftant rectangle, which the reduced equation will fupply; taking the line AD pofitive or negative, according to it's value which the fame equation will give, and the point A either to the right or left of the afymptote TT, according as the laft term of the *homogeneum* fhall be pofitive or negative, as is clear by Fig. 66, 67, 68, 69.

The conftant term ab has hitherto been taken for pofitive, but if it were negative it could make no other alteration, but to make negative the conftant rectangle of the reduced equations, which cafe has already been conftructed. Wherefore the firft of the four equations propofed has now been conftructed in general.

As to the fecond equation of thofe exhibited above, which is $-\frac{gxx}{h} + xy = ab \pm mx \pm ny$; the firft fubftitution to be made is $y - \frac{gx}{h} = z$, that is, $y = z + \frac{gx}{h}$, and let all the reft be done as before.

Therefore, to obtain the ordinate y, it will be neceffary to join $\frac{gx}{h}$ to z, whence in each cafe of Fig. 66, 67, 68, 69, the triangle ABC muft be defcribed under the line NN, as is feen at AbC, with the fides A$b = h$, bC $= g$, and with the angle AbC equal to the angle which ought to be contained by the co-ordinates of the equation propofed; whence, Ab being produced both ways, and taking any line A$q = x$, the correfponding line qR will be the ordinate y required.

The two laft equations of the four were thefe, but with their figns changed.

$$-\frac{gxx}{h} + xy = -ab \mp mx \mp ny.$$

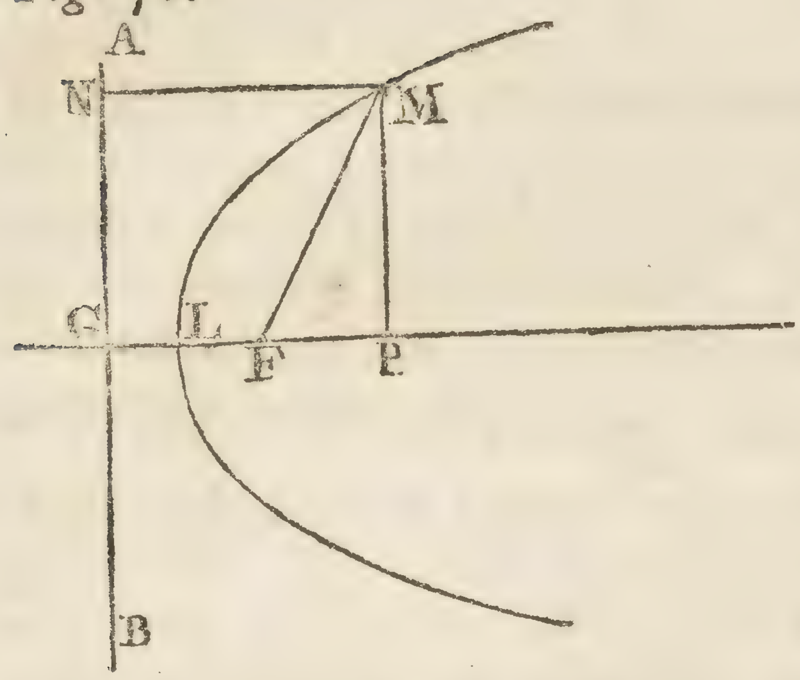
$$\frac{gxx}{h} + xy = -ab \mp mx \mp ny.$$

But this has been already conftructed in the conftruction of the firft, and the other is already conftructed in the conftruction of the fecond; fo that the four equations at firft propofed are now conftructed in general, as was required to be done.

PRO-

PROBLEM I.

Fig. 70.



132. The indefinite right line AB is given in poſition, and the point F is given out of it; it is required to find the *locus* of all the points M, ſuch that, drawing from each of them two right lines, one perpendicular to AB, the other to the point F, theſe two lines may always be equal to each other. A geometrical problem, conſtructed by the parabola.

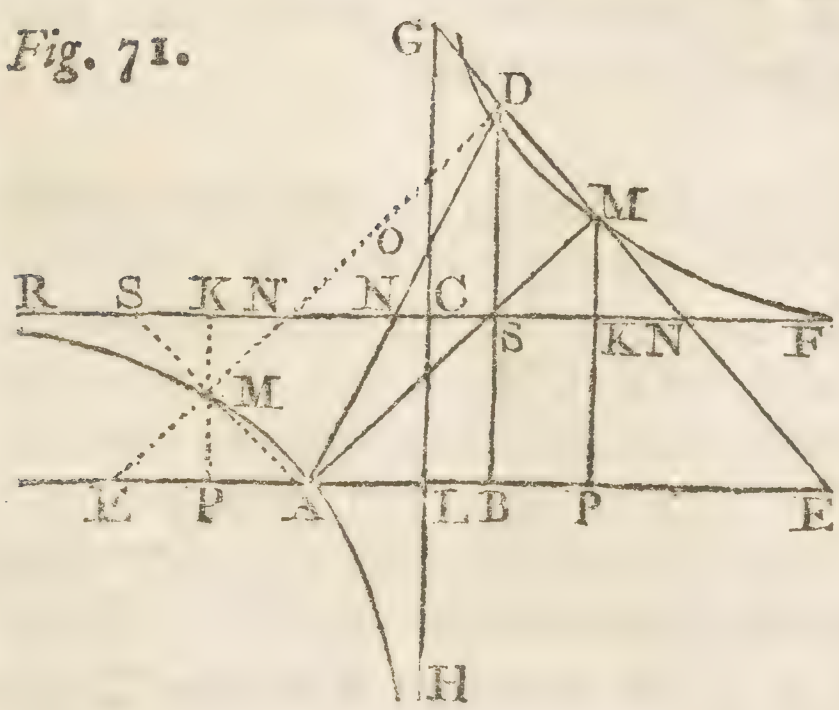
Let M be one of the points required, and let the right lines be drawn, MF to the given point F, and MN perpendicular to BA. Theſe therefore ought to be equal to each other by the condition of the Problem; and therefore, drawing FG perpendicular to AB, and calling it $= a$, let MP be drawn perpendicular to it, and make GP $= x$, PM $= y$, it will be PF $= x - a$, and therefore FM $= \sqrt{aa - 2ax + xx + yy}$. But FM $=$ MN $=$ GP; then $x = \sqrt{aa - 2ax + xx + yy}$, that is, $xx = xx - 2ax + aa + yy$, or $2ax - aa = yy$. And making the ſubſtitution of $x - \frac{1}{2}a = z$, it will be $2az = yy$, an equation to the common parabola.

Take GL equal to half GF, and with vertex L, and parameter $= 2a$, deſcribe the parabola LM. This ſhall be the *locus* required, in which taking any line LP $= z$, it will be PM $= y$. But GL $= \frac{1}{2}a$; therefore GP $= z + \frac{1}{2}a = x$, and therefore GP, PM, will be the co-ordinates of the equation propoſed.

It is known from the property of the parabola, that AB is the *directrix*, and F the *focus* of the curve.

PROBLEM II.

Fig. 71.



133. The indefinite right line PAP being given in poſition, and two fixed points A, D, one in the ſame line, and the other out of it; the *locus* is required of all the points M, ſuch that, drawing the lines MA to the given point A, and DME from the given point D through the point M, it may always be AM equal to the portion ME, comprehended between the point M, and the point E, in which the ſame line DME meets the given line PAP. Another, conſtructed by the hyperbola between the aſymptotes.

 From

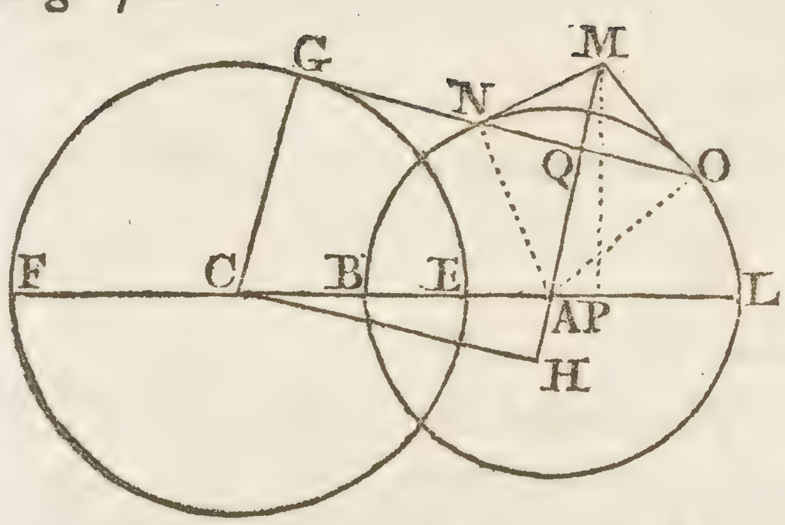
From the given point D, and from the point M, which is ſuppoſed to be one of thoſe required, draw the lines DB, MP, perpendicular to the given line PAP. Then the lines AB, BD, will be known, and therefore make AB $= 2a$, BD $= 2b$, AP $= x$, PM $= y$. Let the right lines AM, DME, be drawn. Now, by the condition of the Problem, AM = ME, and it will be alſo PE = AP $= x$. And becauſe of ſimilar triangles EBD, EPM, it will be EB . BD :: EP . PM. And, ſubſtituting the analytical values, $2x - 2a \,.\, 2b :: x \,.\, y$. Whence the equation $xy - ay = bx$. Make the ſubſtitution of $x - a = z$, it will be $zy = bz + ab$, or $zy - bz = ab$. Make another ſubſtitution of $y - b = p$, and it will be at laſt $pz = ab$, an equation to the hyperbola between the aſymptotes.

On the line PAP given in poſition, from the given point A take AL $= a$, and raiſe LC $= b$ perpendicular to it. Then through the point C drawing the right line RF parallel to PP, between the aſymptotes RF, HG, draw the two oppoſite hyperbolas DM, AM, with the rectangle ab, which ſhall paſs through the points D, A. Taking any line CK $= z$, it will be KM $= p$. But AL $= a$, LC $= b$; therefore AP $= a + z = x$, and PM $= p + b = y$, ſhall be the co-ordinates of the Problem, and the hyperbolas ſhall be the *locus* required.

PROBLEM III.

A problem with three caſes, conſtructed by the parabola, ellipſis, and hyperbola.

Fig. 72.



134. Two circles EGF, BNO, being given, and alſo their centres C, A; if, from any point G of the periphery of the circle EGF, be drawn a tangent GNO, which meets the other circle BNO in the points N, O; and from theſe two points, if we draw two tangents NM, OM, the *locus* of all the points M is required, in which the ſaid tangents meet one another.

From the point M, which is one of thoſe to be found, let be drawn MP perpendicular to CA, and from the centre A draw the right line AM. Becauſe the triangles ANM, AOM, are equal, for the angles at N, O, are right ones, and the ſides AN, NM, are equal to the ſides AO, OM, it will be alſo the angle NMA = OMA; whence in the triangles NMQ, OMQ, becauſe the ſide MQ is common, and MO = MN, it will be QN = QO, and AM perpendicular to NO. From the centre C to the point of contact draw the right line CG, which will be parallel to AM, it being alſo perpendicular to NO. Make AB $= a$, CE $= b$, CA $= c$, AP $= x$, PM $= y$, and therefore AM $= \sqrt{xx+yy}$.

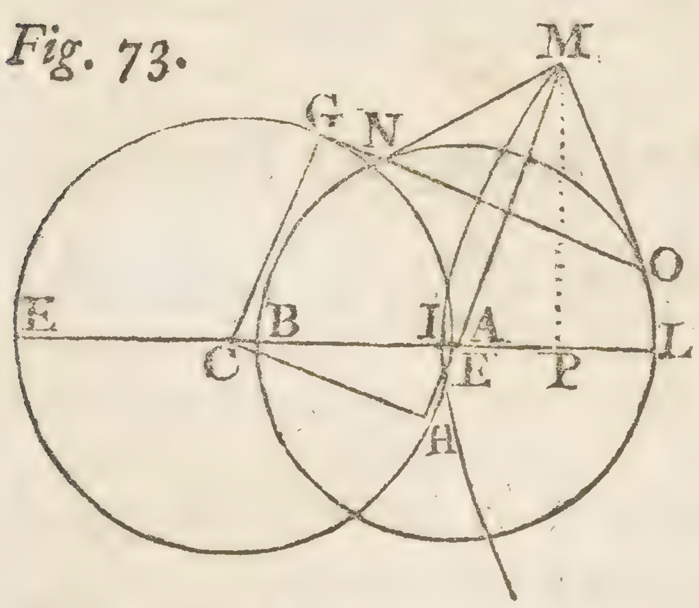
In

In the similar triangles AOM, AQO, it will be AM . OA :: OA . AQ; and substituting the analytical values, we shall find $AQ = \frac{aa}{\sqrt{xx+yy}}$. Draw CH perpendicular to MA, produced if need be; it will be HQ = CG, and therefore $HA = b - \frac{aa}{\sqrt{xx+yy}}$. But the triangles CAH, AMP, will be similar; therefore PA . AM :: AH . AC; that is, $x . \sqrt{xx+yy} :: b - \frac{aa}{\sqrt{xx+yy}} . c$; and multiplying extremes and means, $cx = b\sqrt{xx+yy} - aa$, or $cx + aa = b\sqrt{xx+yy}$. Then squaring, $ccxx + 2aaccx + a^4 = bbxx + bbyy$, that is, $yy + \frac{bb-cc}{bb}xx - \frac{2aacx}{bb} - \frac{a^4}{bb} = 0$.

In this equation there are three cases that ought to be distinguished; that is, when $b = c$, when b is greater than c, and when c is greater than b.

First, let $b = c$, then the equation will be $yy - \frac{2aax}{b} - \frac{a^4}{bb} = 0$, or $yy = \frac{2a^2x}{b} + \frac{a^4}{bb}$. And finding a rectangle $2bf = aa$, put it instead of aa in the last term of the second member, and it will be $yy = \frac{2aax + 2aaf}{b}$; and making the substitution of $x + f = z$, it will be at last $yy = \frac{2aaz}{b}$, an equation to the *Apollonian* parabola. On the right line CA, towards C take $AI = \frac{aa}{2b} = f$, and with vertex I, axis IL, parameter $\frac{2aa}{b}$, let the parabola IM be described. This will be the *locus* required; in which, taking any line IP = z, it will be PM = y; but AI = f, then AP = $z - f = x$, and the lines AP, PM, will be the co-ordinates of the Problem.

Fig. 73.

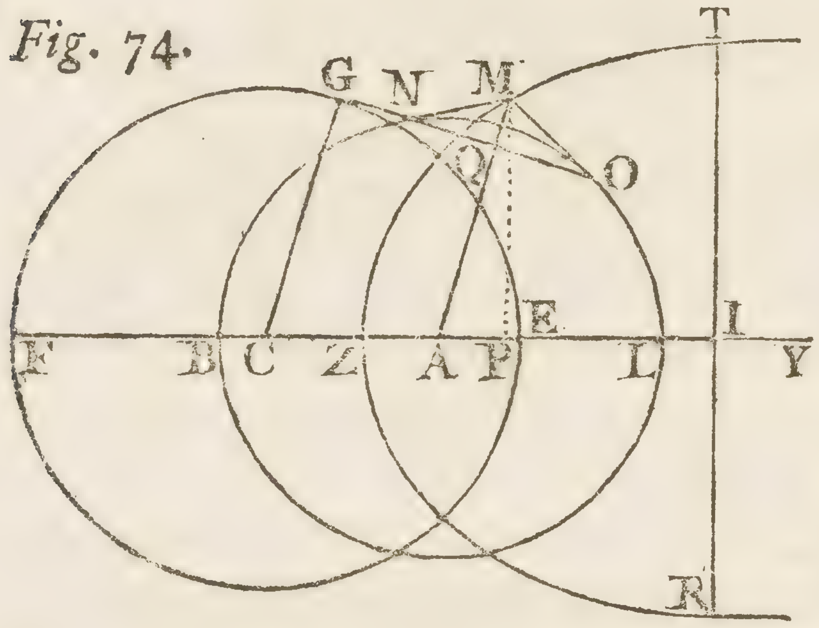


Secondly, let b be greater than c, which will make the term $\frac{bb-cc}{bb}xx$ to be positive. If we write the equation thus, $\frac{bb-cc}{bb}xx - \frac{2aacx}{bb} = \frac{a^4}{bb} - yy$; or thus, $xx - \frac{2aacx}{bb-cc} = \frac{a^4}{bb-cc} - \frac{bbyy}{bb-cc}$, and adding to both members the square $\frac{a^4cc}{\overline{bb-cc}|^2}$, it will be $xx - \frac{2aacx}{bb-cc} + \frac{a^4cc}{\overline{bb-cc}|^2} = \frac{a^4bb}{\overline{bb-cc}|^2} - \frac{bbyy}{bb-cc}$; and

 making

making the fubftitution of $x - \frac{aac}{bb - cc} = z$, it will be finally $\frac{bbyy}{bb - cc} = \frac{a^4bb}{\overline{bb - cc})^2} - zz$, which is an equation to the ellipfis.

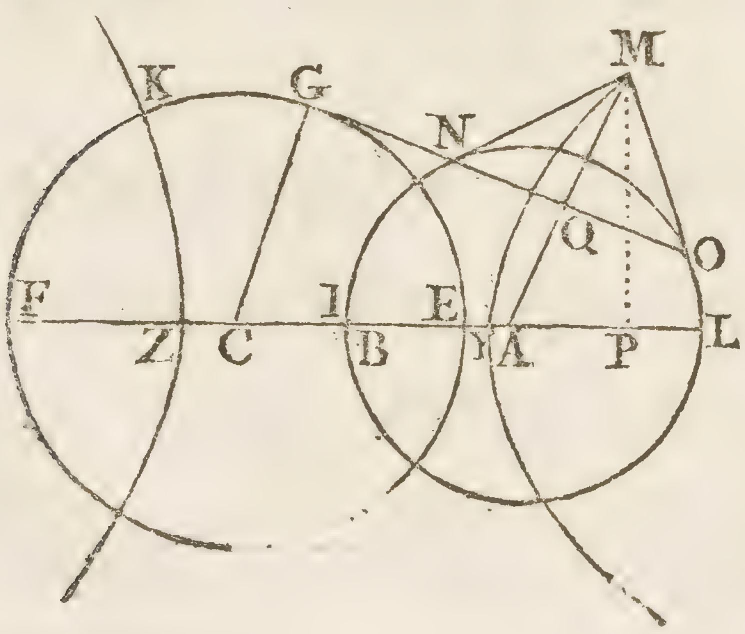
Fig. 74.



From the point A towards Y take the portion AI $= \frac{aac}{bb - cc}$, and with centre I, tranfverfe axis ZY $= \frac{2aab}{bb - cc}$, and conjugate RT $= \frac{2aa}{\sqrt{bb - cc}}$, let the ellipfis RZTY be defcribed, which will be the *locus* required. In this, taking any line IP $= - z$, (that is, on the negative fide,) and it will be PM $= y$. But AI $= \frac{aac}{bb - cc}$; therefore AP $= z + \frac{aac}{bb - cc} = x$, and therefore the lines AP, PM, will be the co-ordinates of the Problem.

Laftly, let c be greater than b, then the quantity $\frac{bb - cc}{bb}xx$ will be negative, and therefore the equation is $\frac{cc - bb}{bb}xx + \frac{2aacx}{bb} = yy - \frac{a^4}{bb}$, or $xx + \frac{2aacx}{cc - bb} = \frac{bbyy - a^4}{cc - bb}$. Add the fquare $\frac{a^4cc}{\overline{cc - bb})^2}$ on both fides, and the equation will be $xx + \frac{2aacx}{cc - bb} + \frac{a^4cc}{\overline{cc - bb})^2} = \frac{bbyy}{cc - bb} + \frac{a^4bb}{\overline{cc - bb})^2}$. And making the fubftitution of $z = x + \frac{aac}{cc - bb}$, it will be at laft $zz - \frac{a^4bb}{\overline{cc - bb})^2} = \frac{bbyy}{cc - bb}$, an equation to an hyperbola, when referred to it's axis.

Fig. 75.



On the right line CA, towards the point C take the portion AI $= \frac{aac}{cc - bb}$, and with centre I, tranfverfe axis ZY $= \frac{2aab}{cc - bb}$, and conjugate $= \frac{2aa}{\sqrt{cc - bb}}$, defcribe the oppofite hyperbolas YM, ZK; thefe fhall be the *locus* required. In which, taking any line IP $= z$, it will be PM $= y$. But AI $= \frac{aac}{cc - bb}$; then AP

$AP = z - \frac{aac}{cc - bb} = x$. And therefore the lines AP, PM, will be the coordinates of the Problem.

In this Problem it is always ſuppoſed, that the circle EFG is greater than the circle BNO, or that b is greater than a; but if it ſhould be either $b = a$, or $b \angle a$, the *locus* of the points required in the firſt caſe would always be a parabola, in the ſecond an ellipſis, and in the third two oppoſite hyperbolas; ſo that it would be needleſs to diſtinguiſh theſe caſes, which make no variation in the *loci*.

PROBLEM IV.

Fig. 76.

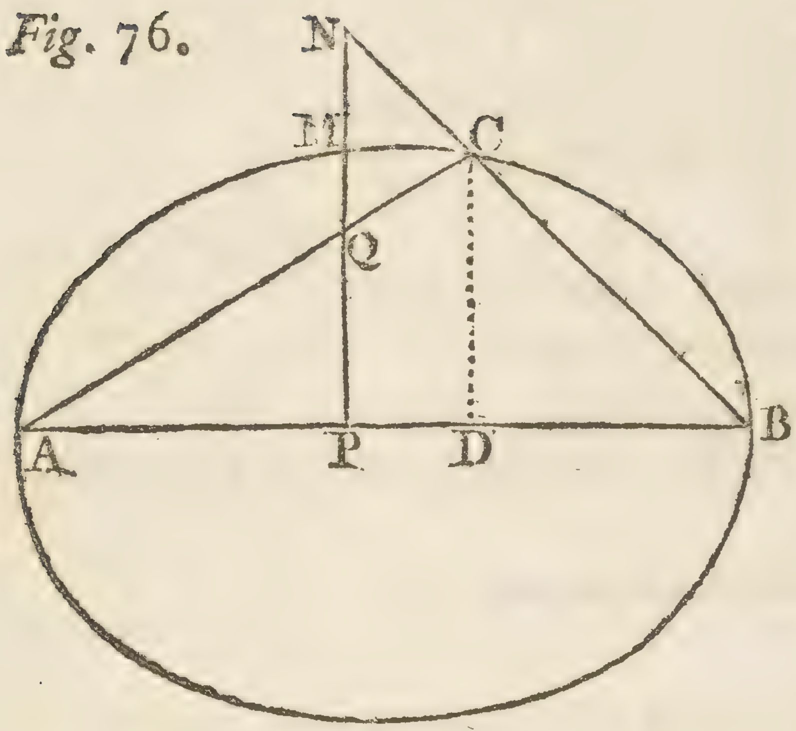
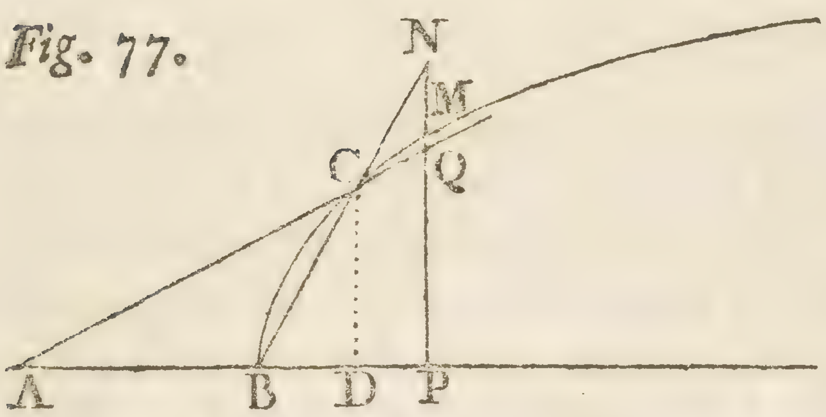


Fig. 77.



135. Two right lines AC, CB, (Fig. 76, 77.) are given in poſition on the right line AB, which cut one another in C; the *locus* is required of all the points M, ſuch that, drawing through them a perpendicular PMN to AB, which cuts the line AC in the point Q, and the line BC in the point N, the ſquare of PM may be equal to the rectangle PQ × PN. A *locus* to the Conic Sections conſtructed.

Let the right line CD be drawn parallel to PM; this will fall either between the points A, B, as in Fig. 76, or on one ſide of them, as in Fig. 77.

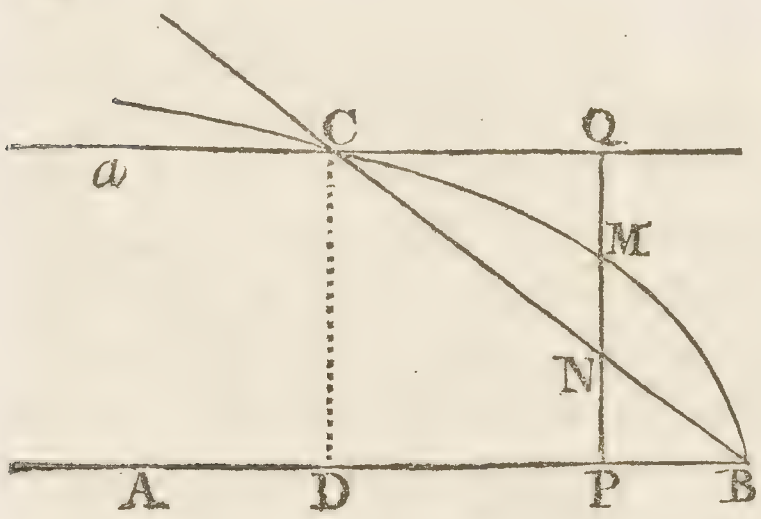
Firſt, let it fall between the points A, B, and make $AB = a$, $AP = u$, $PQ = x$, $PM = y$, $PN = z$. By the condition of the Problem, it will be $zx = yy$. But the ratio of AP to PQ is given, which therefore may be put as m to n. Alſo, the ratio of BP to PN is given, which may be as b to c. Then it will be $PQ = x = \frac{un}{m}$, and $PN = z = \frac{ac - uc}{b}$. Theſe values therefore being ſubſtituted in the equation $zx = yy$, it will be $yy = \frac{ac - uc}{b} \times \frac{un}{m}$, or $\frac{bmyy}{cn} = au - uu$, an equation to an ellipſis with tranſverſe axis $AB = a$, conjugate $a\sqrt{\frac{cn}{bm}}$. Such an ellipſis AMB being deſcribed, the upper half AMCB will be the *locus* required.

Now

Now let the point D (Fig. 77.) fall on one side of the points A, B, and make, as above, $AB = a$, $AP = u$, $PM = y$, $PQ = x$, $PN = z$; it will be $BP = u - a$, and therefore $PN = \frac{uc - ac}{b}$. But, by the condition of the Problem, $zx = yy$, and $x = \frac{un}{m}$, as before. Therefore, making a substitution of the values of z and x, it will be $yy = \frac{uc - ac}{b} \times \frac{un}{m}$, or $\frac{bmyy}{cn} = uu - au$, an equation to the hyperbola.

At the vertex B, with the transverse axis $= a$, and the conjugate axis $= a\sqrt{\frac{cn}{bm}}$, describe the hyperbola BCM; this will be the *locus* required.

Fig. 78.

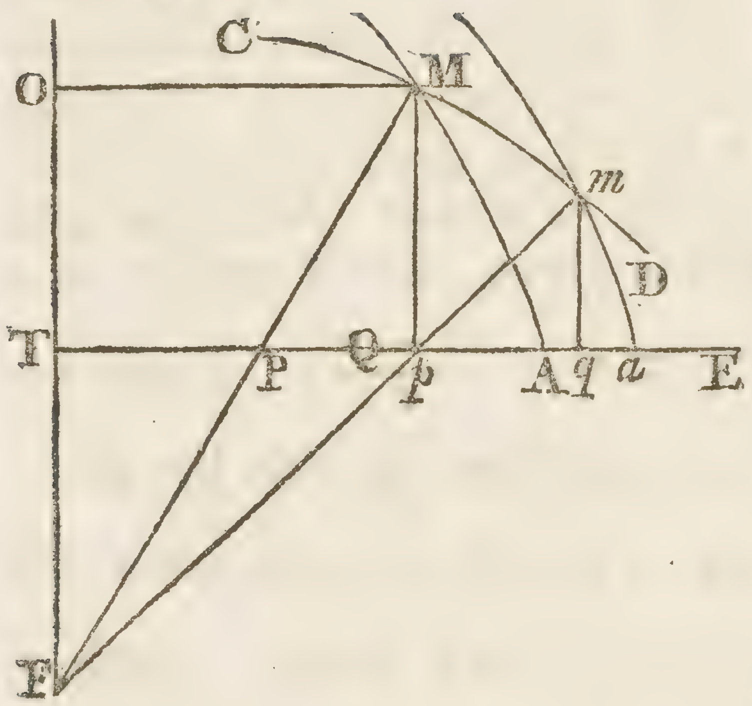


If the right line AC should not fall upon AB, but should be parallel to it, as it would be in the position aC, AB, the right line PQ would be given; therefore, making $PQ = m$, $AB = a$, $BP = u$, $PN = z$, $PM = y$, and supposing $BP \,.\, PN :: m \,.\, n$, the equation $xz = yy$ would become $yy = un$. Wherefore, with vertex B, axis AB, parameter $= n$, describe the *Apollonian* parabola BMC, and this would be the *locus* required in this case.

PROBLEM V.

Another.

Fig. 79.



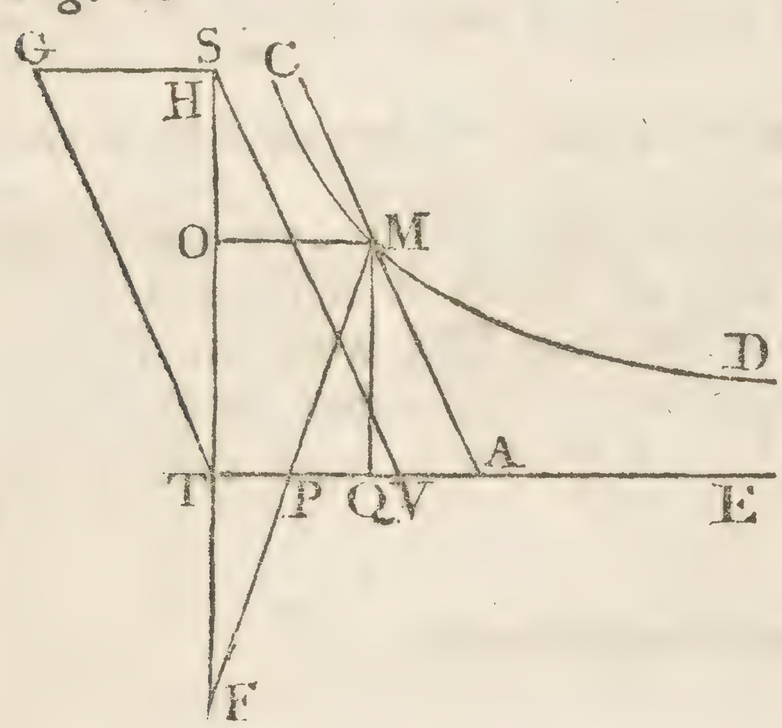
136. Let there be a curve AM, the equation of which is given, and let it's axis be the right line AT, out of which let there be a fixed point F, from whence let be drawn the right line FM, which cuts the curve in the point M, and the axis in the point P. Now the right line FM, moving about the point F, causes the whole plane AMP to move parallel to itself upon the line ET, the point P being fixed in respect of the point A, but moveable upon the axis TA, that is, AP being a given line. In the mean while, the point M will describe a curve CMD. It is required to know what kind of curve this is.

Let

Let the curve be now arrived at the point a of the right line ET; it will be, by the conftruction of the Problem, Pp = Aa, and therefore AP = ap. Make AP = a, FT = b; and from the point M letting fall the perpendicular MQ to ET, make TQ = x, QM = y, AQ = t. Becaufe of the fimilar triangles FOM, PMQ, it will be FO . OM :: QM . PQ, that is, $b + y \,.\, x :: y \,.\,$ PQ $= \frac{xy}{b+y}$. But PQ $= a - t$; therefore $\frac{xy}{b+y} = a - t$, or $xy = ab - bt + ay - ty$.

Now, in this canonical equation, if we fubftitute the value of t given by y, and by the known quantities of the equation of the curve AM, we fhall have the required equation of the curve CMD.

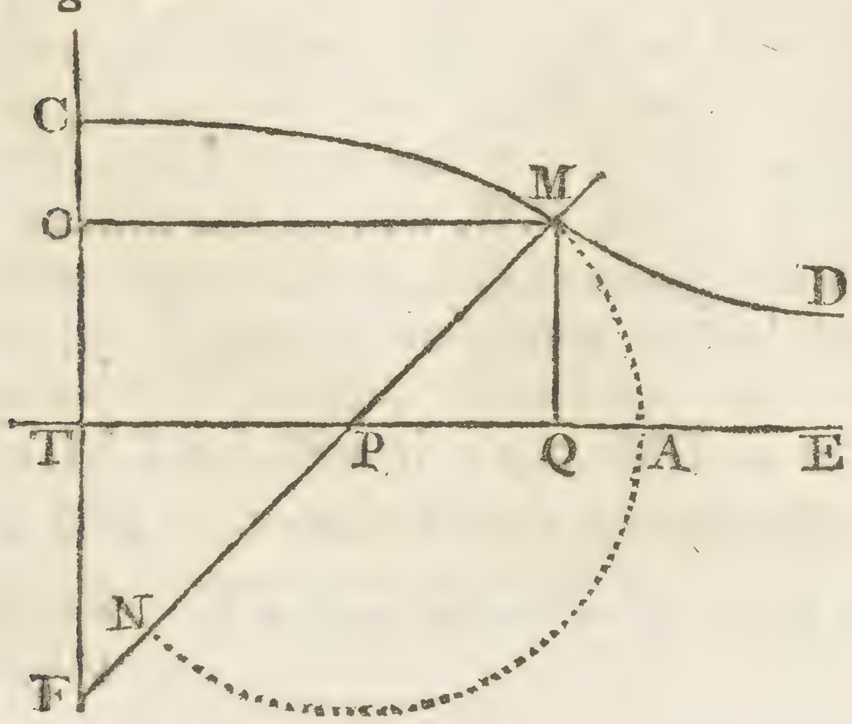
Fig. 80.



Firft, let AM be a right line. The ratio of t to y will be given, which let be that of m to n; then $t = \frac{my}{n}$. And, fubftituting this value of t in the canonical equation, it will be $\frac{myy}{n} = ab - xy - \frac{bmy}{n} + ay$; a *locus* to the hyperbola between the afymptotes.

To conftruct it in the given figure, on FO take any portion TH, and in a right angle draw HG fuch, that it may be FH . HG :: $n \,.\, m$; draw TG, and upon TA taking the portion TV $= \frac{an - bm}{n}$, from the point V draw VS parallel to TG; and between the afymptotes VS, VE, defcribe the hyperbola CMD with the conftant rectangle $= \frac{abg}{n}$; (making the known line TG = g.) Then taking any abfcifs TQ = x, the correfponding ordinate will be QM = y, and the hyperbola will be the *locus* of the equation $\frac{myy}{n} = ab - xy - \frac{bmy}{n} + ay$.

Fig. 81.

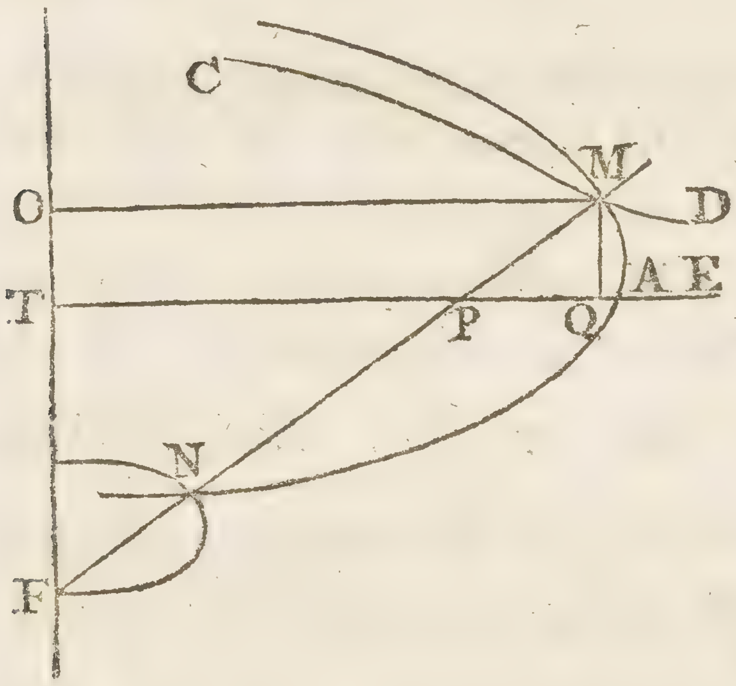


In the fecond place, let AM be a circle defcribed with centre P, radius AP = a. By the property of the circle, it will be AQ $= t = a - \sqrt{aa - yy}$; and inftead of t fubftituting this value in the general equation, it will be $xy = \overline{b + y} \times \sqrt{aa - yy}$, an equation to the conchoid of *Nicomedes*. And the curve CMD, which is defcribed by the interfection M of the right line FM with the fuperior arch of the circle AM, will be the upper conchoid, ET

S will

will be the asymptote, F the pole. And the curve which is generated by the intersection N of the right line FM with the circle under ET, will be the lower conchoid. This appears evidently from the nature of the conchoid, and from the condition of the Problem. For the two lines PM, PN, intercepted between the asymptote and the curve, will always be equal to the radius of the circle AP.

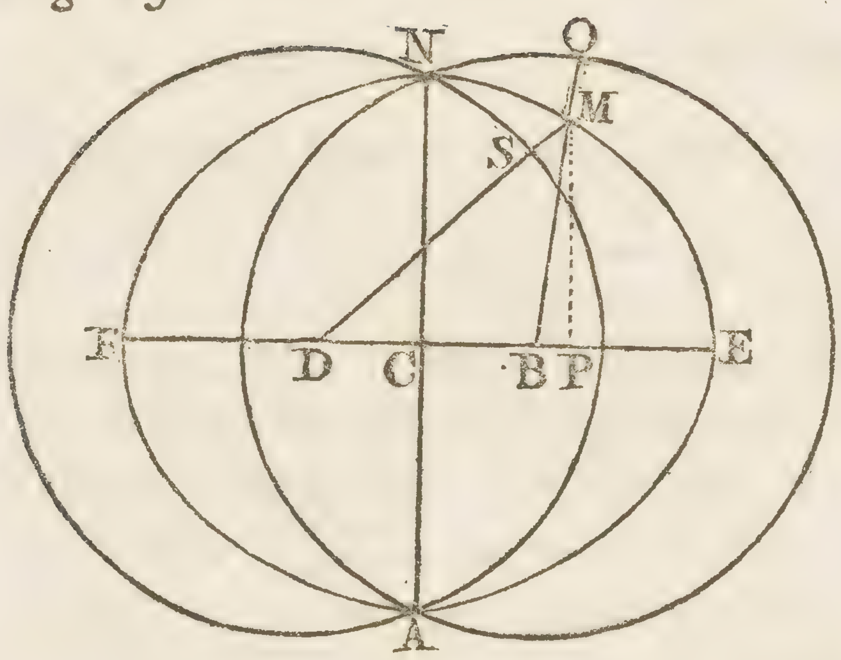
Fig. 82.



In the third place, let the curve AM be an *Apollonian* parabola, with a parameter AP $= a$. On this hypothesis, it will be $t = \frac{yy}{m}$; and this value of t being substituted in the canonical equation, it will be $xy - ay + \frac{y^3}{m} = ab - \frac{byy}{m}$, that is, $y^3 + mxy + byy - amy - abm = 0$. This is an equation to two parabolical conchoids, one of which is described by the intersection of the line FM with the superior part of the parabola; the other by the intersection with the inferior part. And the right line ET will in this case be the asymptote of the curve.

PROBLEM VI.

Another. *Fig.* 83.



137. Two equal circles being given, cutting each other in two points A, N, and their centres D, B, being given; it is required to find the *locus* of all the points M such, that their distances from the said circles may always be equal to one another.

Let M be one of the points required; then drawing from the centres D, B, through this point the right lines DM, BO, then MS, MO, will be the distances from the given circles, which ought to be equal by the condition of the Problem. Therefore make DS $=$ BO $= a$, DB $= b$, and the perpendicular MP being let fall upon DB produced, make DP $= x$, PM $= y$; it will be DM $= \sqrt{xx + yy}$, and SM $= \sqrt{xx + yy} - a$. But BP $= x - b$, therefore BM $= \sqrt{xx - 2bx + bb + yy}$, and thence OM $= a - \sqrt{xx - 2bx + bb + yy}$. But it ought

ought to be SM = MO; whence we ſhall have the equation $\sqrt{xx + yy} - a = a - \sqrt{xx - 2bx + bb + yy}$. By the methods already taught this will be reduced to $xx - bx + \frac{1}{4}bb = aa - \frac{4aayy}{4aa-bb}$; and making the ſubſtitution of $x - \frac{1}{2}b = z$, it will be $zz = aa - \frac{4aayy}{4aa-bb}$, or $\frac{4aayy}{4aa-bb} = aa - zz$, which is an equation to an ellipſis.

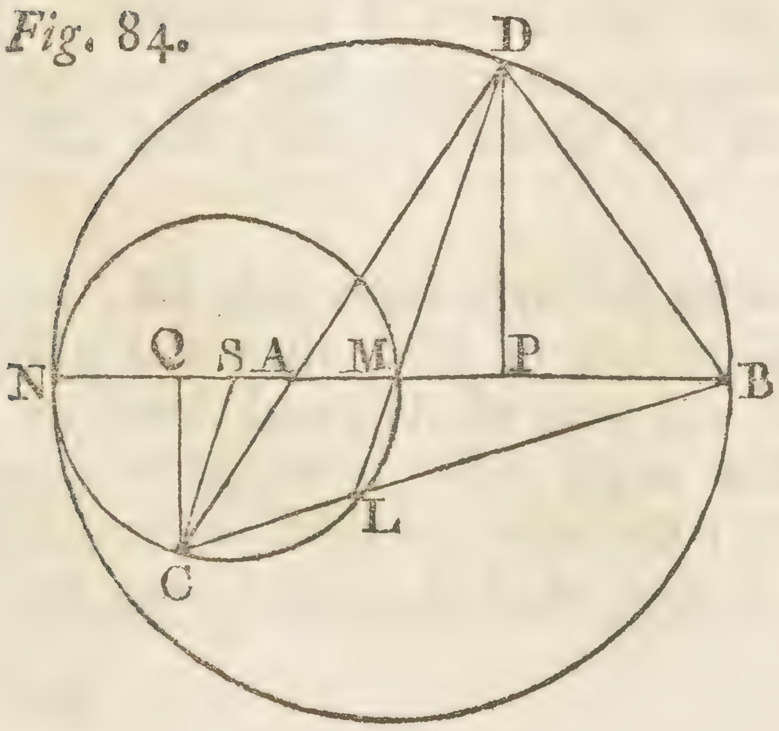
Let the right line DB be biſected in the point C, and with centre C, tranſverſe axis FE $= 2a$, and conjugate AN $= \sqrt{4aa - bb}$, let the ellipſis FAEN be deſcribed, which will be the *locus* required. For, taking any line CP $= z$, it will be PM $= y$; but CD $= \frac{1}{2}b$, therefore DP $= z + \frac{1}{2}b = x$, and therefore the lines DP, PM, are the co-ordinates of the Problem propoſed.

It would be needleſs to diſtinguiſh the caſes, in which a is greater, equal to, or leſs than b, becauſe the Problem will ſtill be of the ſame nature, b being always leſs than $2a$, as plainly appears.

It follows from this conſtruction, that the points D, B, will be the *foci* of the ellipſis, and that it's conjugate axis will be terminated at the points, in which the two circles cut each other. And firſt, becauſe DS = BO, and SM = MO, it will be DS + SM + MB, that is, DM + MB = 2DS; but 2DS = FE, therefore, by the known property of the ellipſis, the points D, B, will be it's *foci*. This ſuppoſed, by another property of the ellipſis relating to the *foci*, conceiving the lines BA, BN, to be drawn, it will be BN = BA = CE. But this is verified in the points, in which the two given circles will cut each other; for D, B, are their centres, and CE, by conſtruction, is equal to the ſemidiameter of the ſame circles. Therefore the ellipſis will paſs through the ſaid points of interſection of the given circles. Q. E. D.

PROBLEM VII.

Fig. 84.



138. The right line AB being given, to find the *locus* of ſuch points D, that, in the produced line DA, taking AC half of AD, and drawing to the point B the right line CB, this may be equal to CD. Another.

Let D be one of the points required, from whence let fall DP perpendicular to AB. Make AB $= a$, AP $= x$, PD $= y$; it will be AD $= \sqrt{xx + yy}$, and, by the condition of the Problem, AC $= \frac{1}{2}\sqrt{xx + yy}$: where-

 fore

fore $CD = CB = \frac{1}{2}\sqrt{xx+yy}$. From the point C draw CQ perpendicular to BA produced. Now, becaufe of the fimilar triangles AQC, APD, and $AD = 2AC$, it will be $AP = 2AQ$, and $PD = 2QC$; whence $CQ = \frac{1}{2}y$, and $AQ = \frac{1}{2}x$. Therefore $BQ = a + \frac{1}{2}x$. Now $CBq = CQq + BQq = aa + ax + \frac{1}{4}xx + \frac{1}{4}yy$. But $CBq = CDq = \frac{9}{4} \times \overline{xx+yy}$; whence we fhall have the equation $\frac{9}{4}xx + \frac{9}{4}yy = aa + ax + \frac{1}{4}xx + \frac{1}{4}yy$, which is reduced to $xx - \frac{1}{2}ax = \frac{1}{2}aa - yy$. Now, adding to both members the fquare $\frac{1}{16}aa$, and making the fubftitution of $x - \frac{1}{4}a = z$, it will be finally $zz = \frac{9}{16}aa - yy$, an equation to the circle.

Therefore, taking $BM = \frac{3}{4}a$, and with centre M, and radius BM, defcribe the circle NDB, this will be the *locus* required; in which, taking any line $MP = z$, it will be $PD = y$; but $AM = \frac{1}{4}a$, therefore $AP = z + \frac{1}{4}a = x$, and the lines AP, PD, will be the co-ordinates of the propofed Problem.

If we would have alfo the *locus* of the points C, this would be another Problem of a like nature, which might be refolved in the following manner.

Make $AQ = p$, $QC = q$, which is perpendicular to BN; it will be $AP = 2p$, $PD = 2q$; but $AM = \frac{1}{4}a$, and $MB = \frac{3}{4}a$. Then $NA = \frac{1}{2}a$, and therefore $NP \times PB = \frac{1}{2}aa + ap - 4pp$. But, by the property of the circle, $NP \times PB = PDq$ and $= 4qq$. Then it will be $4qq = \frac{1}{2}aa + ap - 4pp$. Whence $\frac{1}{8}aa - qq = pp - \frac{1}{4}ap$. Add to both fides the fquare $\frac{1}{64}aa$, and making the fubftitution of $p - \frac{1}{8}a = t$, it will be $qq = \frac{9}{64}aa - tt$. Whence, with diameter $MN = \frac{3}{4}a$ defcribing the femicircle NCM, this will be the *locus* of all the points C; in which, taking from the centre S any line $SQ = t$, it will be $QC = q$. But $AS = \frac{1}{8}a$ by the conftruction. Then $AQ = t + \frac{1}{8}a = p$, and the lines AQ, QC, will be the co-ordinates of the Problem.

Thefe two Problems may be demonftrated conjunctly in form of a theorem, after the following manner.

In the given line AB is taken MB equal to $\frac{3}{4}$ of AB, and with centre M, radius MB, a circle NDB is defcribed; and alfo with diameter MN the circle NCM; through the point A drawing any how the right line CD terminated at the periphery of each circle, and from the point C the right line CB to the extremity of the diameter, it will always be DA the double of AC, and CD equal to CB.

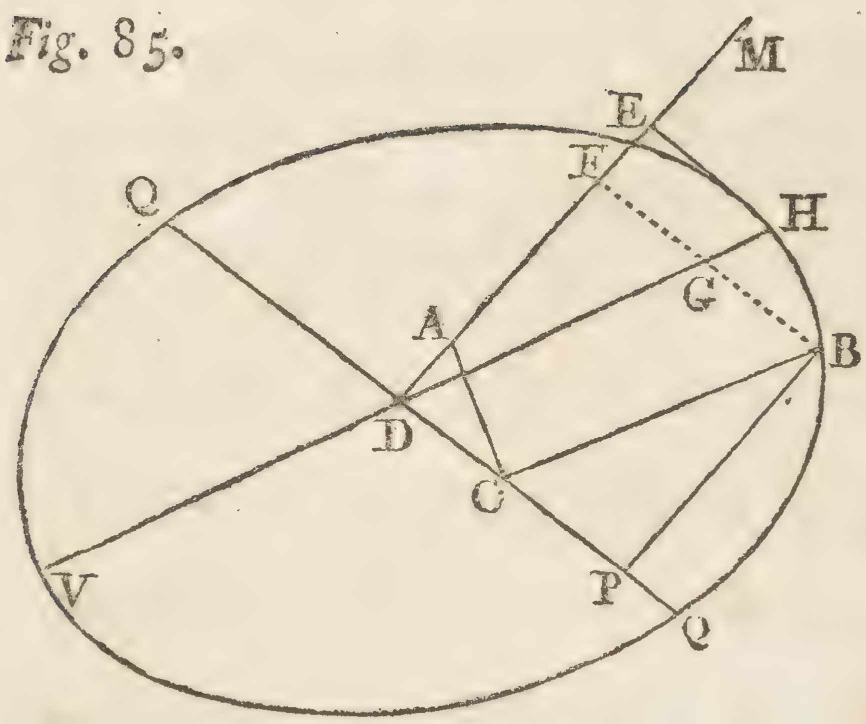
Let S be the centre of the circle NCM, and let the right lines SC, DL, be drawn through the centres S, M. Becaufe SM is half of MB, then will SM be $\frac{3}{8}$ of AB. But AM is $\frac{1}{4}$ of it; therefore SA will be $\frac{1}{8}$ of AB, and therefore $\frac{1}{2}$ of AM. But SC is alfo half of DM, and the angle SAC is equal to the angle DAM; therefore it is eafy to perceive, that the triangle SAC is fimilar to the triangle DAM, and that therefore AC is half of AD, which was the firft thing.

But

But if the triangles SAC, ADM, be ſimilar, then the angle SCA will be equal to the angle ADM; whence the right lines SC, DL, will be parallel, and conſequently the triangles BLM, BCS, are ſimilar, and therefore ML will be the fourth proportional to BS, SC, and MB. But BS $= \frac{9}{8}$AB, SC $= \frac{3}{8}$AB, MB $= \frac{6}{8}$AB. Therefore ML $= \frac{2}{8}$AB $=$ AM. But MD $=$ MB, and the angle AMD $=$ LMB. Therefore the triangles AMD, BML, are equal, and the angle ADM $=$ MBL. But alſo the angle MDB $=$ MBD, ſo that the angle CDB $=$ CBD, and therefore the ſide CB $=$ CD; which was the ſecond thing.

PROBLEM VIII.

Fig. 85.



139. The two ſides AC, CB, of the *norma* ACB being given, the *locus* is required of all the points, through which the extremity B of the ſide CB will paſs, whilſt the *norma* moves in ſuch manner, that it's point A ſhall always be upon the line DM, and the point C upon the line DP, which is ſuppoſed perpendicular to DM. Another.

From the point B let fall BP perpendicular to DP, and make DP $= x$, PB $= y$, AC $= a$, CB $= b$; it will be CP $= \sqrt{bb-yy}$, DC $= x - \sqrt{bb-yy}$. But the angles DCA, BCP, taken together, are equal to a right angle, as alſo the angles BCP, CBP; and therefore the angles DCA, CBP, will be equal to each other. Then the triangles ADC, BCP, will be ſimilar, and it will be AC . CD :: BC . BP, that is, $a \,.\, x - \sqrt{bb-yy} :: b \,.\, y$, and thence $ay = bx - b\sqrt{bb-yy}$; and, by ſquaring and ordering, the equation will be $xx - \frac{2axy}{b} + \frac{aayy}{bb} = bb - yy$. Make the ſubſtitution of $x - \frac{ay}{b} = z$, and we ſhall have the equation $zz = bb - yy$, which is to the ellipſis.

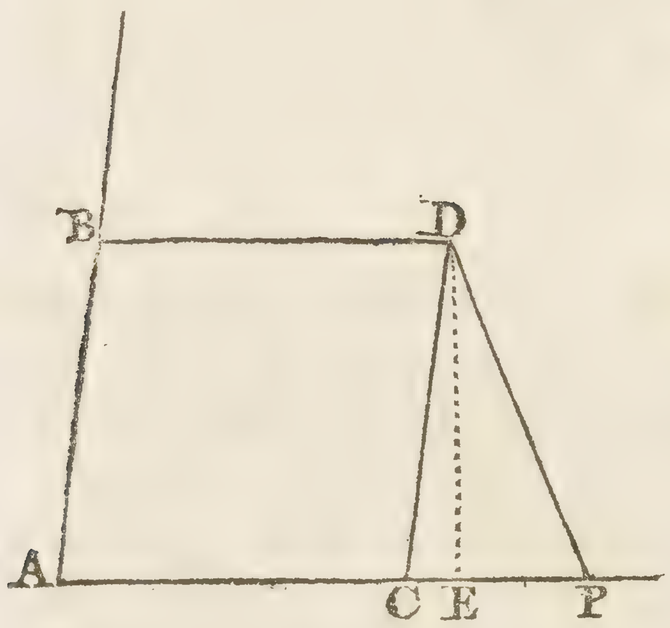
On the indefinite line DM deſcribe the triangle DEH with it's ſides DE $= b$, EH $= a$, and with the right angle DEH, becauſe the co-ordinates of the Problem make a right angle; and let the known line DH $= f$. With tranſverſe ſemidiameter DH $= f$, and with the conjugate ſemidiameter DQ $= b$ and parallel to EH, deſcribe the ellipſis HBQ; it ſhall be the *locus* required. For, taking any line DF $=$ PB $= y$, it will be GB $= z$, FG $= \frac{ay}{b}$; there-

fore

fore $FB = z + \frac{ay}{b} = x = DP$. And therefore the lines DP, PB, are the co-ordinates of the Problem.

PROBLEM IX.

Another. *Fig.* 86.

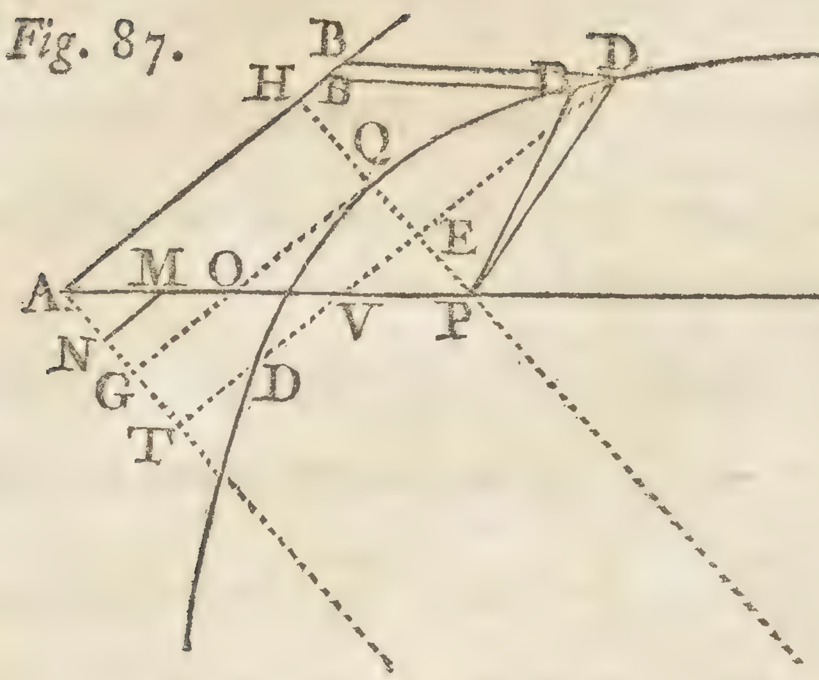


140. The angle BAP being given, and the point P being alſo given; it is required to find the *locus* of all ſuch points D, that, drawing the two right lines, BD parallel to AP, and DP to the given point P, the lines BD, DP, may always be to each other in the given ratio of d to e.

Drawing DC parallel to AB, make $AP = a$, $AC = x$, $CD = y$, $CP = a - x$. Becauſe the angle BAP or DCE is given, drawing DE perpendicular to AP, the ratio of CD to CE is given, which may be $CD . CE :: d . b$; then $CE = \frac{by}{d}$, $AE = x + \frac{by}{d}$, $EP = a - x - \frac{by}{d}$, or elſe $= x + \frac{by}{d} - a$, $PD = \frac{ex}{d}$. Then it will be $CDq - CEq = DPq - PEq$, that is, $yy = \frac{eexx}{dd} - aa - xx + 2ax + \frac{2aby}{d} - \frac{2bxy}{d}$, or $yy + \frac{2bxy}{d} + \frac{bbxx}{dd} = \frac{ee + bb - dd}{dd} xx + 2ax - aa + \frac{2aby}{d}$, by adding the ſquare $\frac{bbxx}{dd}$ on both ſides. But here it may be obſerved, that the quantity $ee + bb - dd$ may either be equal to, greater, or leſs than, nothing; and, firſt, let it be equal to nothing, in which caſe the equation will become $yy + \frac{2bxy}{d} + \frac{bbxx}{dd} = \frac{2aby}{d} + 2ax - aa$. And making the ſubſtitution of $y + \frac{bx}{d} = z$, it will be $zz - \frac{2abz}{d} = 2ax - \frac{2abbx}{dd} - aa$. Then adding $\frac{aabb}{dd}$ on both ſides, it will be $zz - \frac{2abz}{d} + \frac{aabb}{dd} = 2ax - \frac{2abbx}{dd} + \frac{aabb - aadd}{dd}$. Now, making the ſubſtitution of $z - \frac{ab}{d} = p$, it will be $pp = \frac{2addx - 2abbx + aabb - aadd}{dd}$, or $pp = \overline{x - \frac{1}{2}a} \times \frac{2add - 2abb}{dd}$; and making $x - \frac{1}{2}a = q$, it will become at laſt $pp = \frac{2add - 2abb}{dd} q$, an equation to the *Apollonian* parabola.

Let

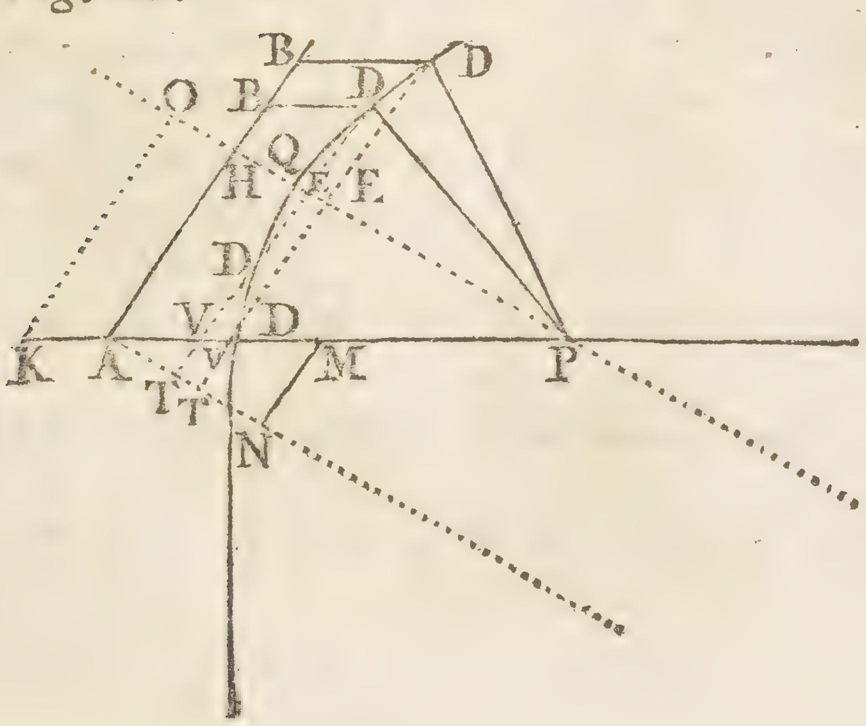
Fig. 87.



Let BAP be the given angle; the given line, AP $= a$. On AP, produced indefinitely, let there be defcribed the triangle AMN with the angle AMN $=$ BAP; and let AM . MN :: $d . b$. Produce AN indefinitely, and in AB take AH $= \frac{ab}{d}$, and draw HE indefinitely, and parallel to AN. Bifect AP in O, and draw OQ parallel to AB. With vertex Q, on the diameter QE, with parameter $= \frac{2add - 2abb}{df}$, (making $f =$ AN,) and with the ordinates parallel to AB, defcribe the parabola QD. Take any line QE $= x$, it will be ED $= y$, and this parabola will be the *locus* required.

In the fecond place, let $ee + bb - dd$ be greater than nothing, or a pofitive quantity. Affuming therefore the equation, and making $ee + bb - dd = hh$, it will be $yy + \frac{2bxy}{d} + \frac{bbxx}{dd} = \frac{hhxx}{dd} - aa + 2ax + \frac{2aby}{d}$. And making the fame fubftitution of $y + \frac{bx}{d} = z$, it will be $zz - \frac{2abz}{d} = \frac{hhxx}{dd} - aa + 2ax - \frac{2abbx}{dd}$; and adding $\frac{aabb}{dd}$, and making the fubftitution of $z - \frac{ab}{d} = p$, it will be $ddpp = hhxx + 2addx - 2abbx - aadd + aabb$; that is, $xx + \frac{2ad^2x - 2ab^2x}{hh} = \frac{ddpp}{hh} + \frac{aadd - aabb}{hh}$; make $\frac{add - abb}{hh} = m$, then $xx + 2mx = \frac{ddpp}{hh} + am$; and adding mm to each fide, it will be $xx + 2mx + mm = \frac{ddpp}{hh} + am + mm$, and making $x + m = q$, it will be finally $qq = \frac{ddpp}{hh} + am + mm$, that is, $qq - am - mm = \frac{ddpp}{hh}$, an equation to an hyperbola.

Fig. 88.

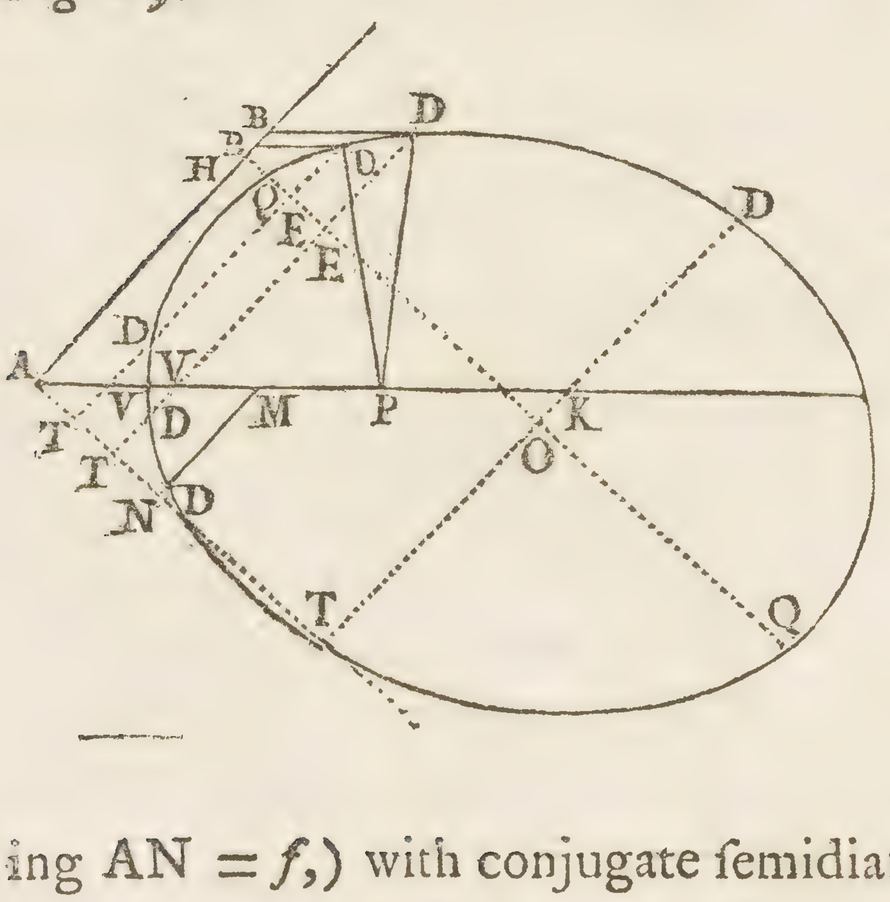


Let BAP be a given angle; the given line, AP$=a$. Upon AP, produced indefinitely, let the triangle AMN be defcribed with the angle AMN equal to BAP; and let it be AM . MN :: $d . b$. Produce AN indefinitely, and in AB take AH $= \frac{ab}{d}$, and through the point H draw the indefinite line OE parallel to AN. Then make AK $= m$, and draw KO parallel to AH. With centre O,

O, tranfverfe femidiameter $OQ = \frac{f\sqrt{am+mm}}{d}$, and conjugate femidiameter $= \frac{b\sqrt{am+mm}}{d}$, parallel to AH, (by f is denoted the known line AN,) let the hyperbola QD be defcribed. Then taking any line $AV = x$, it will be $VD = q$, and this hyperbola will be the *locus* required.

Laftly, let $ee + bb - dd$ be lefs than nothing, that is negative. Make then $ee + bb - dd = -hh$, and making $y + \frac{bx}{d} = z$, the equation will be $zz - \frac{2abz}{d} = -\frac{hhxx}{dd} - aa + 2ax - \frac{2abbx}{dd}$; and adding $\frac{aabb}{dd}$ on both fides, it will be $zz - \frac{2abz}{d} + \frac{aabb}{dd} = -\frac{hhxx}{dd} + \frac{2addx - 2abbx}{dd} + \frac{aabb - aadd}{dd}$; and making the fubftitution of $z - \frac{ab}{d} = p$, it will be $ddpp = -hhxx + 2addx - 2abbx + aabb - aadd$, that is, $xx + \frac{2abbx - 2addx}{hh} = \frac{aabb - aadd}{hh} - \frac{ddpp}{hh}$. Make $\frac{add - abb}{hh} = m$, and we fhall have $xx - 2mx = -am - \frac{ddpp}{hh}$, and adding mm on both fides, $xx - 2mx + mm = mm - am - \frac{ddpp}{hh}$; laftly, making the fubftitution of $x - m = q$, it is $\frac{ddpp}{hh} = mm - qq - am$, an equation to an ellipfis.

Fig. 89.



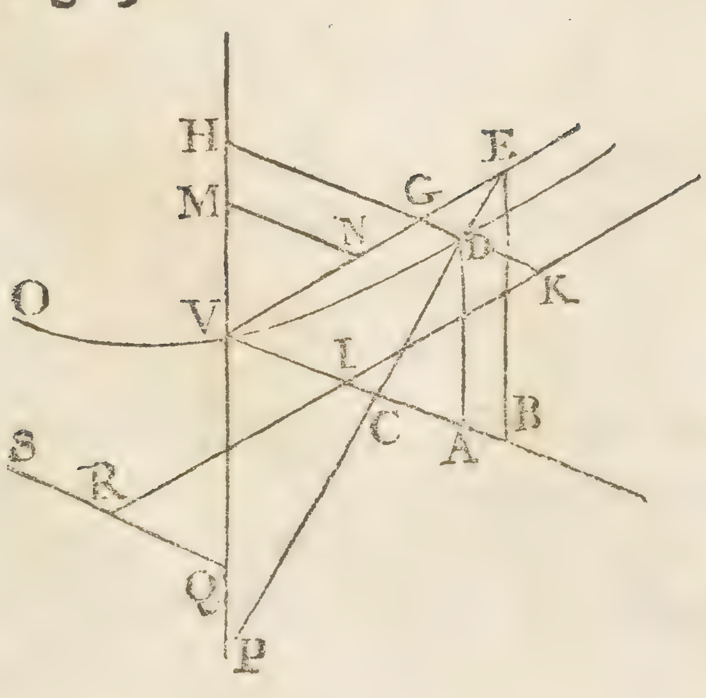
Let BAP be the given angle, and the given line $AP = a$. On AP, indefinitely produced, defcribe the triangle AMN with the angle AMN equal to BAP. Make AM . MN :: d . b, and produce AN indefinitely, and in AB take $AH = \frac{ab}{d}$, and through the point H draw the indefinite line HE parallel to AN. On AP produced take $AK = m$, which in this cafe is always greater than $AP = a$, and draw KO parallel to AB. With centre O, tranfverfe femidiameter $OQ = \frac{f\sqrt{mm - am}}{d}$ (making $AN = f$,) with conjugate femidiameter $= \frac{b\sqrt{mm - am}}{d}$, and parallel to AH, defcribe the ellipfis QD. Then taking any line $AV = x$, it will be $VD = y$; and this fhall be the *locus* required.

141. I said above that AK $= m$ was greater than AP $= a$; in relation to which I think it necessary to explain how we may know which of two complicate quantities is the greater. Let there be made between them a comparison of majority or minority, as you please, and then proceed as in an equation, by transposing, dividing, &c. and making other operations, till you arrive at a known consequence; which, if it be true either absolutely or hypothetically, the comparison that was made will be absolutely or hypothetically true; but if false, this will likewise be false. So, if we desire to know whether m, that is, $\frac{add - abb}{dd - bb - ee}$, be greater than a, or not, make the comparison or supposition $\frac{add - abb}{dd - bb - ee} > a$, and reducing to a common denominator, it will be $add - abb > add - abb - aee$, and expunging the terms that destroy each other, it will be $0 > -aee$; which is very true, for nothing is greater than a negative quantity. Therefore it was true that $\frac{add - abb}{dd - bb - ee}$ was greater than a. A method to determine majority and minority in complicate quantities.

Thus, to know if $aa + 2ab$ be greater than bb, suppose $aa + 2ab > bb$, and add to each side the square bb. It will be $aa + 2ab + bb > 2bb$, and extracting the root, it is $a + b > \sqrt{2bb}$, or $a > \sqrt{2bb} - b$. But, because the quantities a, b, are given, it may always be known whether a be greater than $\sqrt{2bb} - b$, or not. And if it should be so, then also $aa + 2ab$ would be greater than bb. The manner is the same in cases more compounded, and therefore I shall insist on it no longer.

PROBLEM X.

Fig. 90.



142. Two right lines VB, VE, being given in position, and also the point P, about which as a pole the right line PE revolves; to find the *locus* of all the points D, such that it may always be CD to DE in a given ratio. A geometrical problem.

Draw VP, and parallel to it the right lines AD, BE, and let the ratio of CD to DE, or rather of CD to EC, be as d to e; and the angles EVB, EBV, being given, let it be EB to BV as e to h.

 Make

Make VP $= a$, VA $= x$, AD $= y$; it will be EB $= \frac{ey}{d}$, and therefore VB $= \frac{hy}{d}$. Becaufe of the fimilar triangles CVP, CDA, it will be DA . PV :: CA . CV; and, by compounding, DA + PV . PV :: CA + CV . CV; that is, $a + y \,.\, a :: x \,.$ CV, and therefore CV $= \frac{ax}{a+y}$. Again, becaufe of fimilar triangles PVC, EBC, it will be PV . VC :: EB . BC, that is, $a \,.\, \frac{ax}{a+y} :: \frac{ey}{d}$. BC; whence BC $= \frac{exy}{ad + dy}$, and therefore the equation BC + CV = BV, that is, $\frac{exy + adx}{ad + dy} = \frac{hy}{d}$, or $yy - \frac{exy}{h} = \frac{adx}{h} - ay$.

To conftruct this, make $y - \frac{ex}{h} = \frac{ez}{h}$, and, by fubftitution, it will be $\frac{ezy}{h} = -ay - \frac{adz}{h} + \frac{ady}{e}$, that is, $zy + \frac{ahy}{e} - \frac{adhy}{ee} = -\frac{adz}{e}$.

Again, make $z + \frac{ah}{e} - \frac{adh}{ee} = p$; then it will be $py = \frac{aadh}{ee} - \frac{aaddh}{e^3} - \frac{adp}{e}$. And making a third fubftitution of $y + \frac{ad}{e} = q$, it will be $pq = \frac{aaehd - aahdd}{e^3}$, an hyperbola between the afymptotes, the conftant rectangle of which is pofitive, becaufe e will always be greater than d.

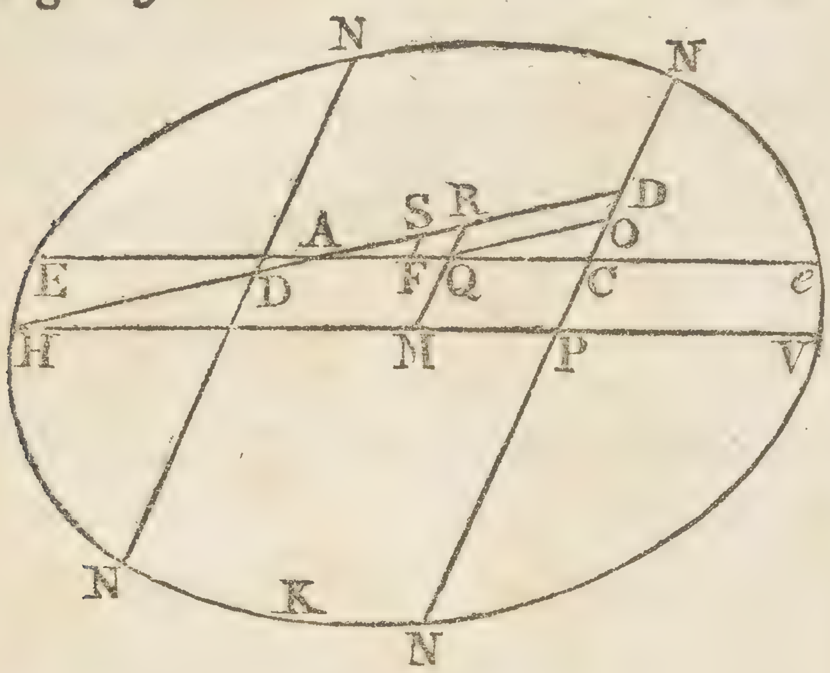
Let PV be produced indefinitely, and take VQ $= \frac{ad}{e}$. From the point Q draw the indefinite line QS parallel to VB, and, taking any point M in the right line PH, draw MN parallel to VB. Then, becaufe of fimilar triangles VMN, EBV, it will be VM . MN :: $e \,.\, h$. Make VI $= \frac{aeh - adh}{ee}$, and through the point I drawing the indefinite right line RIK parallel to VE, between the afymptotes RS, RK, defcribe the hyperbola OVD with the conftant rectangle $= \frac{aaehh - aaddh}{e^3} \times \frac{f}{e}$, (making the known line VN $= f$,) which will neceffarily pafs through the point V. Taking any line VH $= y$, it will be HD $= x$, that is, VA $= x$, AD $= y$, and the curve thus conftructed is the *locus* of the points D.

A fpecimen of the demonftration of thefe examples.

143. We may obferve here, that the equations expreffing the properties of the curves defcribed in thefe Examples, or Problems, ought to be the fame with the equations propofed to be conftructed, when the operations are truly performed; and therefore may ferve as a demonftration of the method itfelf. This

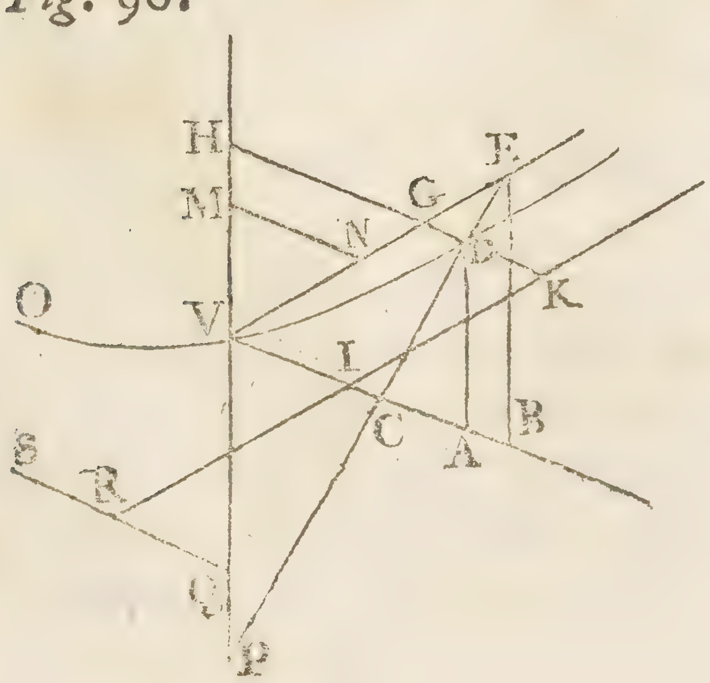
This I have purpofely omitted to do, to avoid being too prolix. However, to give a fhort fpecimen of it, I fhall take the conftructions of Example XIII. and of Problem VIII.

Fig. 65.



And, firft, for the example. Having made AD $= x$, and it being AS $= 2a$, AF $= f$, it will be AC $= \frac{fx}{2a}$, and therefore AR $= \frac{hn}{2m}$; it will be AQ $= \frac{hfn}{4am}$, and thence QC $= \frac{fx}{2a} - \frac{hfn}{4am} =$ MP. Therefore, the femidiameter being HM $= \frac{ef}{2a}$, we fhall have HP $= \frac{ef+fx}{2a} - \frac{hfn}{4am}$, and PV $= \frac{ef}{2a} - \frac{fx}{2a} + \frac{hfn}{4am}$. Thus, becaufe DN $= y$, CD $= \frac{bx}{2a}$, CP $=$ QM $= \frac{1}{2}c$, it will be PN $= y + \frac{bx}{2a} + \frac{1}{2}c$. But, by the property of the ellipfis, it muft be HP $\times$ PV . PNq :: HV . parameter $= \frac{4aem}{fn}$. Thence we fhall have the equation $\frac{eeff-ffxx}{4aa} + \frac{ffhnx}{4aam} - \frac{ffhhnm}{16aamm}$ into $\frac{4aam}{ffn} = \frac{1}{4}cc + \frac{bcx}{2a} + \frac{bbxx}{4aa} + cy + \frac{bxy}{a} + yy$. And, inftead of ee, reftoring it's value $\frac{ccmm + 4agmn + nnhh}{4mm}$, it will be $\frac{1}{4}cc + ag - \frac{mxx}{n} + hx = \frac{1}{4}cc + \frac{bcx}{2a} + \frac{bbxx}{4aa} + cy + \frac{bxy}{a} + yy$. And laftly, reftoring the values of $-\frac{m}{n} = \frac{bb-4aa}{4aa}$, and $h = \frac{bc-2al}{2a}$, we fhall have $ag - xx - lx = cy + \frac{bxy}{a} + yy$, which is the very equation propofed to be conftructed.

Fig. 90.



In the conftruction of the laft Problem it was $\frac{aacdh - aaddh}{e^3} \times \frac{f}{e}$ the conftant rectangle of the hyperbola, and VI $= \frac{aeh - adh}{ee}$, and parallel to the afymptote RS. Alfo, it will be RI $= \frac{adf}{ce}$. But, becaufe of fimilar triangles VMN, VHG, it is VM . VN :: VH . GV, and

therefore

therefore $GV = \frac{fy}{e} = IK$. Then $RK = \frac{adf}{ee} + \frac{fy}{e}$. But $HG = \frac{hy}{e}$, $GK = VI = \frac{aeh - adh}{ee}$. Whence $HK = \frac{hey + aeh - adh}{ee}$. But $HD = VA = x$; then it will be $KD = \frac{hey + aeh - adh - eex}{ee}$, and therefore, by the property of the curve, the rectangle $RK \times KD$ ought to be equal to the constant rectangle, or $\frac{adf + efy}{ee} \times \frac{hey + aeh - adh - eex}{ee} = \frac{aaedh - aaddh}{e^3} \times \frac{f}{e}$. That is, $yy - \frac{exy}{h} + ay - \frac{adx}{h} = 0$, as it ought to be.

If the fame care and induftry were ufed in every Example and Problem, it would fufficiently prove the method of folution to be juft.

SECT. IV.

Of Solid Problems and their Equations.

What are the roots of equations.

144. Any one of thofe quantities is called the Root of an Equation, which, being fubftituted in the equation inftead of that root or letter, according to which the equation is ordered, (or inftead of that letter which reprefents the unknown quantity,) fhall make all the terms of the equation to vanifh or become nothing. Or, which is the fame thing, the root of an equation is each of the feveral values of the unknown quantity, or of that letter which performs the office of an unknown quantity in the equation.

Therefore the roots of the equation $xx - ax + bx - ab = 0$ will be two, one of which is a, the other $-b$; for each of thefe, being fubftituted inftead of x, will make the terms of the equation to vanifh; or, becaufe either a or $-b$ are the values of the letter x in the propofed equation. The roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$ will be 1, 2, 3, or -5; becaufe any one of thefe numbers, being fubftituted inftead of x, will make all the terms to vanifh, and therefore any one of them is the root, or value of the unknown quantity x. The roots of the equation $x^4 - bbxx - aabb - a^4 = 0$ will be $+\sqrt{-aa}$, $-\sqrt{-aa}$, $+\sqrt{aa+bb}$, $-\sqrt{aa+bb}$; and fo of all others.

145. Again,

145. Again, in another ſenſe, thoſe equations are uſed to be taken for the roots of an equation, which are formed by ſubtracting, one by one, the poſitive values from the unknown quantity, or by adding the negative value, and making them equal to nothing. Therefore, in this ſenſe, the roots of the equation $xx - ax + bx - ab = 0$ will be $x - a = 0$, and $x + b = 0$. Thoſe of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$ will be $x - 1 = 0$, $x - 2 = 0$, $x - 3 = 0$, and $x + 5 = 0$. And ſo of others. And, in this ſenſe, it is ſaid, that every equation is the product of all it's roots, becauſe, being continually multiplied into one another, they will exactly produce the given equation, or that of which they are the roots. Hence it is, that the roots of an equation will be ſo many, including alſo the imaginary roots, as is the degree to which the equation ariſes. And therefore a quadratick equation will have two roots, a cubick equation three roots, a biquadratick four roots; and ſo on.

Or otherwiſe, the ſeveral diviſors of an equation.

If $x + a = 0$ be multiplied into $x + b = 0$, there will ariſe the quadratick equation (I.)

$$\begin{array}{lll} xx & + ax & + ab = 0. \\ & + bx & \end{array}$$

And if this again be multiplied into $x - c = 0$, there will ariſe the cubick equation (II.)

$$\begin{array}{llll} x^3 & + ax^2 & + abx & - abc = 0. \\ & + bx^2 & - acx & \\ & - cx^2 & - bcx & \end{array}$$

And if this again be multiplied into $x + d = 0$, it will produce the biquadratick equation (III.)

$$\begin{array}{lllll} x^4 & + ax^3 & + abx^2 & - abcx & - abcd = 0. \\ & + bx^3 & - acx^2 & + abdx & \\ & - cx^3 & + adx^2 & - acdx & \\ & + dx^3 & - bcx^2 & - bcdx & \\ & & + bdx^2 & & \\ & & - cdx^2 & & \end{array}$$

Thus, if $x + \sqrt{ab} = 0$ be multiplied into $x - \sqrt{ab} = 0$, the product will be $xx - ab = 0$; and if this be multiplied into $x + c = 0$, it will make $x^3 + cx^2 - abx - abc = 0$; and again, if this be multiplied into $x + c = 0$, it will make

$$\begin{array}{lllll} x^4 & + 2cx^3 & - abx^2 & - 2abcx & - abcc = 0. \\ & & + ccx^2 & & \end{array}$$

If $x + \sqrt{-ab} = 0$ be multiplied into $x - \sqrt{-ab} = 0$, and then into $x + a = 0$, it will produce the cubick equation $x^3 + ax^2 + abx + aab = 0$.

146. Therefore, if we had the means of knowing what were the values of all, or of any of the unknown quantities of an equation, we might always divide it by ſo many ſimple equations as are thoſe known values, by adding the negative values to the unknown quantity, and ſubtracting the poſitive. Whence the firſt equation before will be diviſible by $x + a$, and by $x + b$. The ſecond,

Equations might be reſolved by diviſion, if their roots were known.

fecond, by $x + a$, $x + b$, and $x - c$. The third, by $x + a$, $x + b$, $x - c$, $x + d$. By this, compound equations will be reduced to fo many fimple equations as is the number of the roots, if all be known; or may be depreffed by fo many degrees as is the number of the known roots, if they be not all known. So that, for inftance, an equation of the fifth degree may be reduced to one of the fourth, if one of it's roots be known; or to the third, if two roots be known; and fo on.

Hence is known the nature or formation of the feveral co-efficients.

147. From the method by which equations are produced, (which equations are always underftood to be reduced to nothing, and in which the greateft term in refpect of the unknown quantity, or in refpect of that letter by which the terms are ordered, muft be pofitive and free from a co-efficient,) it is eafy to perceive that the co-efficient of the unknown letter, or that by which the equation is ordered, in the fecond term is the fum of all the roots of the equation affected with contrary figns; the co-efficient of the third term is the fum of all the products of all the pairs of roots which can be formed; the co-efficient of the fourth term is the fum of all the products of all the ternaries or threes; and fo on to the laft or conftant term, which is the product of all the roots multiplied continually into one another.

When the fecond term will be wanting.

148. Hence it may be inferred, that the fum of the pofitive roots muft neceffarily be equal to the fum of all the negative roots, in all fuch equations in which the fecond term is wanting: and that the fum of the pofitive roots is greater than the fum of the negative, when the fecond term is affected with a negative fign; and contrarily, when it is affected with a pofitive fign.

How the abfence of a term is to be denoted.

149. When any term is wanting in an equation, it is ufual to fupply it's place by an afterifm $*$; as in $x^4 * + aaxx - b^3x + a^4 = 0$, the fecond term is wanting. In $x^4 - ax^3 * - b^3x + a^4$, the third term is wanting; and fo in others.

Surd roots and imaginary roots always proceed by pairs.

150. If an equation have no term affected by an imaginary quantity, either it's roots fhall be all real, or, if it have any imaginary roots, they fhall always be even in number, and equal two by two; only with this difference, that one muft be affirmative and the other negative. For, becaufe the fecond term is the fum of all the roots, if this be prefent in the equation, when the imaginary roots do not deftroy one another, two by two, with contrary figns, fome imaginary root muft neceffarily be in the co-efficient, which is contrary to the fuppofition. Now, if the fecond term be wanting, it muft needs follow, that the fum of the pofitive roots is equal to the fum of the negative, and confequently the fum of the pofitive imaginary roots muft be equal to the fum of the negative imaginary roots, otherwife they cannot deftroy one another in the manner aforefaid. Wherefore equations, whofe degree is an odd number, will neceffarily have one real root at leaft; and thofe of an even degree may have all

all their roots imaginary or impoſſible. For the ſame reaſon, we may make like concluſions about ſurd roots. That is to ſay, if the equation have no ſurd or irrational terms in it, it's roots will either be all rational, or the irrational roots will be in even numbers, and will be equal two by two, but with contrary ſigns.

151. There are equations which have all their roots poſitive, others have all their roots negative, others have both poſitive and negative. So ſome have all their roots imaginary, others have all real, and laſtly, others have both real and imaginary. Various rules are given by writers of Algebra, to determine in any given equation the number of poſitive and negative roots, alſo of real and imaginary roots. But, becauſe theſe rules and their demonſtrations are very perplexed and prolix, and of but little uſe, I ſhall here omit them, thinking it ſufficient to take notice, firſt, that if all the roots be negative, all the terms of the equation will be poſitive. For, in this caſe, ſince all the terms of the ſimple equations are poſitive, that is, of all the roots taken in the ſecond ſenſe, § 145, from whence the propoſed equation is ſuppoſed to be produced, all the products will alſo be poſitive. Secondly, that if all the roots be poſitive, the terms of the equation will be poſitive and negative alternately. For the firſt term will always be poſitive by ſuppoſition. The ſecond term will be negative, becauſe it contains the ſum of all the roots, which being poſitive, will be negative in the ſimple equations. The third term, containing the ternaries or products of all the pairs, will be poſitive. And ſo on. And therefore an equation compoſed of poſitive and negative ſigns alternately, will have all it's roots poſitive.

Affections of the roots how diſtinguiſhed.

Whence, if the terms of an equation ſhall not have all their ſigns poſitive, or ſhall not have them poſitive and negative alternately, there will be both poſitive and negative roots. It ſhall alſo be another ſure proof, that the equation contains both poſitive and negative roots, if there be any term wanting; for no term can be abſent, but that the products of which it is formed muſt deſtroy one another by contrary ſigns; that is, there muſt be both affirmative and negative roots. This obſervation will aſſiſt us in it's proper place, among the many diviſors of the laſt term of an equation, to ſelect thoſe only by which the diviſion is to be attempted. Becauſe, if the equation ſhall have only poſitive roots, it would be of no uſe to try the diviſion by poſitive diviſors; and if it ſhall have only negative roots, it would be needleſs to try by negative diviſors. And the trials muſt be made by each of them, when there are both poſitive and negative roots.

But all this muſt be underſtood of ſuch equations in which all the roots are real; for where there are imaginary roots the rule does not obtain. For example, let the equation be $x^3 + bx^2 + aax + aab = 0$, in which all the terms are poſitive, and yet the roots are one poſitive and two negative, that is, $x = -b$, a real root, and $x = \pm\sqrt{-aa}$, two imaginary or impoſſible roots, one poſitive, the other negative.

152. Equations

Affections of the roots of equations of the third or fourth degree.

152. Equations of the third and fourth degree, in which the second term is wanting, if the third term be affected with the positive sign, will certainly have imaginary roots; for, if all the roots were real, the third term could not but be affected with the negative sign; the reason of which is, that in cubick equations, when the second term is wanting, the sum of the positive roots is equal to the sum of the negative, and therefore either one positive is equal to two negative, or two positive roots are equal to the one negative. Let the three roots, for instance, be represented by a, b, and $-c$, or else by $-a$, $-b$, and $+c$; then the co-efficient of the third term will be $ab - ac - bc$. But, on supposition that the second term is wanting, it will be $a + b = c$. Therefore ac will be greater than ab, and consequently $ab - ac - bc$ will be a negative quantity.

Now, in equations of the fourth degree, there may be either three positive roots and one negative, as $+a$, $+b$, $+c$, and $-d$; or there may be three negatives and one affirmative, as $-a$, $-b$, $-c$, and $+d$; or there may be two negatives and two affirmative, as $-a$, $-b$, $+c$, and $+d$. In the first and second case, the co-efficient of the third term will be $ab + ac + bc - ad - bd - cd$. But, by supposition, it ought to be $a + b + c = d$, so that ad will be greater than ab, cd than ac, bd than bc; and therefore $ad + bd + cd$ will be greater than $ab + ac + bc$, and consequently the third term will be negative. In the third case, the co-efficient of the third term will be $ab - ac - bc - ad - bd + cd$, and it ought to be $a + b = c + d$. Here, if we make $m = a + b = c + d$, it will be $mm = \overline{a+b} \times \overline{c+d} = ac + ad + bc + bd$, and $mm = \overline{a+b}^2 = aa + 2ab + bb$, and also $mm = \overline{c+d}^2 = cc + 2cd + dd$. Therefore it is $ab = \frac{mm - aa - bb}{2}$, and $cd = \frac{mm - cc - dd}{2}$, and $ab + cd = mm - \frac{aa + bb + cc + dd}{2}$. Therefore mm is greater than $ab + cd$, and $ac + ad + bc + bd$ will be greater than $ab + cd$. Whence the co-efficient of the third term will be negative.

The positive roots may be made to become negative, and *vice versâ*.

153. It is always in our power, in any equation, to make all the positive roots to become negative, and the negative to become positive. Nothing more is required to perform this, than to change all the signs which are in even places, that is, in the second, the fourth, the sixth, &c.; the reason of which is, that the second term being the sum of all the roots, in this therefore are the negative with a positive sign, and the positive with a negative sign, as has been plainly seen at § 145. In forming equations, compounded of the products of simple equations, by changing the signs they also will be changed. The other even terms in order are formed from the products of an odd number of roots, that is, the fourth from three, the sixth from five, &c. Wherefore, if they have the positive sign, they will be formed from the product of all the negative roots, or from an even number of positive roots, and an odd number of

of negative roots. And if they have a negative fign, they will be formed from the product of all the pofitive roots, or an even number of negative roots, and an odd number of pofitive roots. Therefore, by changing the figns of all the even terms, the pofitive roots will become negative, and on the contrary.

As to the odd terms in order, they being formed of even products of roots, if they have the pofitive fign, they will be formed either of an even number of negative roots alone, or of an even number of pofitive roots alone, or of an even number of pofitive, or an even number of negative together. Wherefore, by changing thefe reciprocally, the figns of the terms themfelves will not be changed. Now, if they have a negative fign, they will be formed of the product of an odd number of pofitive roots, into an odd number of negative. Wherefore, by thefe alfo reciprocally, the fign of the terms themfelves will not be changed, and therefore they muft be left as they are.

The equation $\begin{array}{llll} x^3 & + ax^2 & + abx & - abc = 0 \\ & + bx^2 & - acx & \\ & - cx^2 & - bcx & \end{array}$ has three roots. Two are negative, *viz.* $-a$, $-b$, or otherwife, $x + a = 0$, $x + b = 0$, and one is pofitive, *viz.* $+c$, or otherwife, $x - c = 0$. By changing the figns of thofe terms which in the order of the equation are even, it will become

$\begin{array}{llll} x^3 & - cx^2 & + abx & + abc = 0, \\ & - bx^2 & - acx & \\ & + cx^2 & - bcx & \end{array}$ and the pofitive roots will be $x - a = 0$, $x - b = 0$, and the negative root will be $x + c = 0$. It is of no moment whether or no any term be wanting, becaufe in this cafe the afterifm fupplies the vacancy, and then the fame rule obtains. Thus, in the equation $x^3 * - 28x + 48 = 0$, the affirmative roots of which are $x - 2 = 0$, $x - 4 = 0$, and the negative root is $x + 6 = 0$. By changing the figns of the even terms in order, it will be $x^3 * - 28x - 48 = 0$, the negative roots of which are $x + 2 = 0$, $x + 4 = 0$, and the affirmative root is $x - 6 = 0$.

The roots of an equation may be increafed or diminifhed at pleafure.

154. Any equation being given, by means of congruous fubftitutions it is eafy to increafe or diminifh all it's roots, though yet unknown, by any given quantity; that is, it may be transformed into another equation, the roots of which fhall be the fame as thofe of the propofed equation, but increafed or diminifhed by fome given quantity. Let the unknown quantity of the equation be put equal to a new unknown quantity, adding or fubtracting the given quantity; adding, if we would have it increafed, or fubtracting, if we would have it diminifhed. Then, in the propofed equation, inftead of the unknown quantity and it's powers, their values muft be fubftituted, expreffed by the other unknown quantity and the given conftant quantity; from whence another equation will arife, the roots of which will be fuch as are required. Let the equation be $x^4 + 4x^3 - 19x^2 - 106x - 120 = 0$, the roots of which we would

would have increaſed by the number 3. Make $x + 3 = y$, whence $x = y - 3$, $x^2 = y^2 - 6y + 9$, $x^3 = y^3 - 9y^2 + 27y - 27$, and $x^4 = y^4 - 12y^3 + 54y^2 - 108y + 81$; therefore, in the propoſed equation, ſubſtituting theſe values inſtead of x and it's powers, it will be transformed into this other equation,

$$\left.\begin{array}{r} y^4 - 12y^3 + 54y^2 - 108y + 81 \\ + 4y^3 - 36y^2 + 108y - 108 \\ - 19y^2 + 114y - 171 \\ - 106y + 318 \\ - 120 \end{array}\right\} = 0; \text{ that is, } y^4 - 8y^3 - y^2 + 8y = 0;$$

and dividing by y, it is $y^3 - 8y^2 - y + 8 = 0$, in which it is plain, that the roots will be greater than the roots of the propoſed equation by the number 3; becauſe it was made $y = x + 3$, and therefore the root y will be equal to every value of x increaſed by 3. And here it may be obſerved, that, in thus increaſing the roots, the poſitive are increaſed by ſuch a quantity, but the negative are diminiſhed by the ſame quantity; for, by adding a poſitive to a negative, if the negative be greater than the poſitive, it will become leſs in it's kind than at firſt; if they be equal, it becomes nothing, if it be leſs, it makes it poſitive. Whence, in the propoſed equation $x^4 + 4x^3 - 19x^2 - 106x - 120 = 0$, the roots of which (though they cannot be found by the methods hitherto taught,) are $+5, -2, -4, -3$, that is, $x - 5 = 0$, $x + 2 = 0$, $x + 4 = 0$, $x + 3 = 0$; one of which is affirmative, the other negative; as I deſired to increaſe them by the number 3, in the transformed equation $y^3 - 8y^2 - y + 8 = 0$, they ought to be $+8, +1, -1$, that is, $y - 8 = 0$, $y - 1 = 0$, $y + 1 = 0$, and are really ſuch. And that which ſhould correſpond to the fourth is $= 0$, becauſe $-3 + 3 = 0$. And, for this reaſon, the reduced equation is only of three dimenſions, though the propoſed equation is of four.

On the contrary, when the roots of an equation are to be diminiſhed by a given quantity, for the ſame reaſon the negative roots are increaſed in their kind by the ſame quantity, but the poſitive may become nothing, if the given quantity be equal to them, and negative if greater. In the ſame equation $x^4 + 4x^3 - 19x^2 - 106x - 120 = 0$, if I ſhould deſire to diminiſh the roots by the number 3, I muſt make $x - 3 = y$, and therefore $x = y + 3$, $x^2 = y^2 + 6y + 9$, $x^3 = y^3 + 9y^2 + 27y + 27$, $x^4 = y^4 + 12y^3 + 54y^2 + 108y + 81$. And therefore, making the ſubſtitutions, the equation will be

$$\left.\begin{array}{r} y^4 + 12y^3 + 54y^2 + 108y + 81 \\ + 4y^3 + 36y^2 + 108y + 108 \\ - 19y^2 - 114y - 171 \\ - 106y - 318 \\ - 120 \end{array}\right\} = 0. \text{ That is, } y^4 + 16y^3 + 71y^2$$

$- 4y - 420 = 0$. And, becauſe the roots of the propoſed equation are $+5, -2, -3, -4$, that is, $x - 5 = 0$, $x + 2 = 0$, $x + 3 = 0$, $x + 4 = 0$; thoſe of the transformed equation ought to be $+2, -5, -6, -7$,

-7, that is, $y-2=0$, $y+5=0$, $y+6=0$, $y+7=0$, as they really are.

Let the equation be $x^3+cx^2-bbx-bbc=0$, and we defire to increafe the roots by a given quantity a. Make $x+a=y$, and therefore $x=y-a$, $x^2=y^2-2ay+aa$, $x^3=y^3-3ay^2+3aay-a^3$. Wherefore, making the fubftitutions, the equation will be

$$\left.\begin{array}{r} y^3-3ay^2+3a^2y-a^3 \\ +\ cy^2-2acy+a^2c \\ -\ bby+abb \\ -\ bbc \end{array}\right\}=0.$$

The roots of this are greater than thofe of the propofed equation by the quantity a. And, in fact, the roots of the propofed equation are $x-b=0$, $x+b=0$, $x+c=0$; but the roots of this are $y-b+a=0$, $y+b-a=0$, and $y+c-a=0$.

155. In like manner, if an equation be given, we may transform it into another, the roots of which are the fame as thofe of the propofed equation, but multiplied or divided by a given quantity, fuppofe f; making a fubftitution of $fx=y$, (x being the unknown quantity of the given equation,) if we would have it multiplied; or of $\frac{x}{f}=y$, if we would have it divided. Thus, alfo, we may make $x=\frac{gy}{f}$, if we defire that the roots of the transformed equation fhould have to thofe of the propofed equation the ratio of f to g. And we may make $\sqrt{fx}=y$, if we would have them to be mean proportionals between the quantity f, and the roots of the propofed equation. In like manner, we may make $x=\frac{1}{y}$, if we defire they may be reciprocals, &c.

Or the roots may be multiplied or divided at pleafure.

156. The reafon of thefe rules is evident. For, affuming the firft cafe, or that of increafing the roots, if we make the fubftitution of $x+a=y$, the values of y extracted from the transformed equation will be equal to $x+a$, or equal to the values of x in the propofed equation increafed by the quantity a. And by a like analogy in the other cafes.

The reafon of thefe operations.

157. Many are the ufes that may be made of thefe fubftitutions; one of which may be, that not having as yet a method of knowing what are the roots of the propofed equation, by transforming it after fome one of the aforementioned manners, we may difcover the roots of the transformed equation; which being increafed, diminifhed, multiplied, divided, &c. by the conftant quantity, according as the fubftitution is made, we fhall alfo know the roots of the propofed equation.

And their ufes.

U 2 158. Another

Equations may be freed from fractions or surds.

158. Another use may be, to free equations, whenever we please, from fractions, and very often from surds. As to fractions, we must make the unknown quantity of the equation equal to some new unknown quantity, divided by the least quantity that is divisible by every one of the denominators of the terms of the equation; which shall be the product of the same, in case that those denominators are prime to each other. Then making the substitutions, and reducing the terms to a common denominator, we shall have another equation which will be free from fractions, the roots of which will be those of the proposed equation, multiplied into the quantity by which the new unknown quantity was at first divided. Let the equation be $y^3 + \frac{1}{6}ay^2 - \frac{1}{3}aby + aab = 0$; if we make $y = \frac{1}{6}z$, $y^2 = \frac{1}{36}z^2$, $y^3 = \frac{1}{216}z^3$, then, by substitution, the equation will become $\frac{z^3}{216} + \frac{az^2}{6\times 36} - \frac{abz}{3\times 6} + aab = 0$. And, reducing to a common denominator, it will be $z^3 + az^2 - 12abz + 216aab = 0$. The roots of this equation divided by 6 will be the roots of the equation proposed.

Let the equation be $x^3 - \frac{ax^2}{b} + \frac{aax}{c} + \frac{a^3}{d} = 0$. Make $x = \frac{z}{bcd}$, and, substituting in the equation, it will be transformed into this, $\frac{z^3}{b^3c^3d^3} - \frac{az^2}{b^3c^2d^2} + \frac{aaz}{bc^2d} + \frac{a^3}{d} = 0$. Then reducing to a common denominator, it will be $z^3 - acdz^2 + a^2b^2cd^2z + a^3b^3c^3d^2 = 0$. Wherefore, if the value of z were known, the value of x would be known also. In like manner, to free equations from surds, we may often proceed thus. Make the unknown quantity equal to a new unknown quantity divided by the radical, and substitute this in the equation. Let the equation be $x^3 - \sqrt{3} \times x^2 + \frac{26}{27}x - \frac{8}{27\sqrt{3}} = 0$. Make $x = \frac{z}{\sqrt{3}}$, and therefore $x^2 = \frac{z^2}{3}$, $x^3 = \frac{z^3}{3\sqrt{3}}$; and, making the substitutions, it will be $\frac{z^3}{3\sqrt{3}} - \frac{z^2\sqrt{3}}{3} + \frac{26z}{27\sqrt{3}} - \frac{8}{27\sqrt{3}} = 0$. Now, multiplying by $3\sqrt{3}$, it will be $z^3 - 3z^2 + \frac{26}{9}z - \frac{8}{9} = 0$. Lastly, by freeing this from fractions after the foregoing manner, that is, making $z = \frac{1}{9}y$, or rather, $z = \frac{1}{3}y$, which in this case will be more compendious, the equation will be $y^3 - 9y^2 + 26y - 24 = 0$. And because, by the first substitution, it is $x = \frac{z}{\sqrt{3}}$, and, by the second, $z = \frac{1}{3}y$, it will be $x = \frac{y}{3\sqrt{3}}$; or the value of x will be equal to the value of y divided by $3\sqrt{3}$.

Let the equation be $x^4 - x^3\sqrt[3]{nn} + px^2\sqrt[3]{n} - qx + \frac{r}{\sqrt[3]{n}} = 0$. Make $x = \frac{y}{\sqrt[3]{n}}$, and therefore $xx = \frac{yy}{\sqrt[3]{nn}}$, $x^3 = \frac{y^3}{n}$, $x^4 = \frac{y^4}{n\sqrt[3]{n}}$; and making the substitutions,

ſtitutions, it will be $\frac{y^4}{n\sqrt[3]{n}} - \frac{y^3\sqrt[3]{nn}}{n} + \frac{pyy\sqrt[3]{n}}{\sqrt[3]{nn}} - \frac{qy}{\sqrt[3]{n}} + \frac{r}{\sqrt[3]{n}} = 0$. And multiplying by $n\sqrt[3]{n}$, it will be $y^4 - ny^3 + npy^2 - nqy + rn = 0$. If we would obſerve the law of homogeneity, equations may be delivered from radicals: but then fractions would thence ariſe, which muſt be reduced as above.

159. Becauſe, by taking away radicals by means of the foregoing ſubſtitutions, nothing elſe is done than multiplying the roots of the equation by that radical, it is eaſy to perceive, that if the radical be quadratick, for example $\sqrt{n}$, it is neceſſary, in order to expunge it out of the equation, that the ſecond term of the equation propoſed ſhall contain $\sqrt{n}$. For, as that term is the aggregate of all the roots of the equation, it muſt be multiplied by $\sqrt{n}$. It will be neceſſary that the third term ſhould not contain $\sqrt{n}$, becauſe, as it is the aggregate of the pairs of the roots of the equation, it muſt be multiplied by the ſquare of $\sqrt{n}$. Thus it will be neceſſary that the fourth ſhould contain $\sqrt{n}$, becauſe, as it is the aggregate of all the ternaries, or products of three roots, it muſt conſequently be multiplied by $n\sqrt{n}$. It will alſo be neceſſary that the fifth term ſhould not contain the radical; and ſo on alternately. For the ſame reaſon, if the radical to be taken away were $\sqrt[3]{n}$, it will be neceſſary, that in the ſecond term of the propoſed equation there ſhould be found $\sqrt[3]{nn}$, in the third $\sqrt[3]{n}$, in the fourth none at all, in the fifth $\sqrt[3]{nn}$, in the ſixth $\sqrt[3]{n}$, in the ſeventh none at all, &c. And the like is to be concluded of other radicals. Conditions for expunging radicals.

160. By means of theſe ſubſtitutions we may alſo take away the ſecond term from any equation. And that will be done by putting the unknown quantity equal to a new unknown quantity, adding or ſubtracting the co-efficient of the ſecond term divided by the index of the degree of the equation given: that is, adding, if the ſecond term have the negative ſign, and ſubtracting, if that ſign be poſitive. Let the equation be $x^2 + ax - bb = 0$; put $x = z - \frac{1}{2}a$, and, by ſubſtitution, it will become Thus the ſecond term of an equation may be taken away.

$$\left.\begin{array}{l} z^2 - az + \frac{1}{4}aa \\ \quad + az - \frac{1}{2}aa \\ \quad\quad - bb \end{array}\right\} = 0. \text{ That is,}$$

$zz * - \frac{1}{4}aa - bb = 0$, or $zz = \frac{1}{4}aa + bb$. Hence it may be ſeen, how all affected quadratick equations may be reſolved more expeditiouſly in this manner, than by that before taught at § 74. Then, only ſubtracting $\frac{1}{2}a$ from the value of z ſo found, we ſhall have the value of x.

Let the equation be $x^3 + bx^2 - abx - a^3 = 0$. Make $x = z - \frac{1}{3}b$, and, by ſubſtitution, it will be

$$\left.\begin{array}{l} z^3 * - \frac{1}{3}bbz + \frac{2}{27}b^3 \\ \quad\quad - abz + \frac{1}{3}abb \\ \quad\quad\quad\quad - a^3 \end{array}\right\} = 0.$$

Whence, taking $\frac{1}{3}b$ from the value of z, we ſhall have the value of x.

Let

Let the equation be $\left.\begin{array}{l}x^4 - 2ax^3 + 2aaxx - 2a^3x + a^4 = 0.\\ - ccxx\end{array}\right.$ Make $x = z + \frac{2a}{4}$, or $x = z + \frac{1}{2}a$. Then, by ſubſtitution, it will be

$$\left.\begin{array}{l} z^4 * + \frac{1}{2}aaz^2 - a^3z + \frac{5}{16}a^4 \\ - ccz^2 - accz - \frac{1}{4}aacc \end{array}\right\} = 0.$$ Then add $\frac{1}{2}a$ to the value of z, and we ſhall have the value of x.

Or the third term may be taken away.

161. And thus we may take away the third term from any equation, proceeding after the following manner.

Let the equation be $x^4 - 3ax^3 + 3aax^2 - 5a^3x - 2a^4 = 0$. Make $x = y - b$, where b is a general quantity, to be determined as occaſion may require. Now, making the ſubſtitutions, it will be

$$\left.\begin{array}{l} y^4 - 4by^3 + 6bby^2 - 4b^3y + b^4 \\ - 3ay^3 + 9aby^2 - 9ab^2y + 3ab^3 \\ + 3aay^2 - 6a^2by + 3a^2b^2 \\ - 5a^3y + 5a^3b \\ - 2a^4 \end{array}\right\} = 0.$$ Now, in this equation, that the third term may be nothing, it is neceſſary that $6bby^2 + 9aby^2 + 3aay^2 = 0$, that is, $b^2 + \frac{3}{2}ab + \frac{1}{2}aa = 0$; and therefore $b = -\frac{3}{4}a \pm \frac{1}{4}a$. Hence we are informed, that the ſubſtitution to be made inſtead of $y - b$, is either $y + \frac{1}{2}a$, or $y + a$; for, indeed, either the one or the other takes away the third term, making the equation $y^4 - ay^3 * - \frac{15}{4}a^3y - \frac{65}{16}a^4 = 0$, or, ſecondly, $y^4 + ay^3 * - 4a^3y - 6a^4 = 0$.

By this artifice it may be known, that, to take away the ſecond term, we muſt make ſuch ſubſtitutions as have been ſhown at § 160.

Or the laſt but one, if the ſecond be wanting.

162. Now if an equation, in which the ſecond term is wanting, is to be transformed into another, in which the laſt term but one ſhall be abſent, it will be ſufficient to ſubſtitute any given quantity, divided by a new unknown quantity, inſtead of the unknown quantity of the equation. Let the equation be $x^4 * + aax^2 - a^3x + a^4 = 0$, and make $x = \frac{aa}{y}$. By ſubſtitution, it will be $\frac{a^8}{y^4} * + \frac{a^6}{y^2} - \frac{a^5}{y} + a^4 = 0$. And reducing this to a common denominator, and dividing by a^4, it will be $y^4 - ay^3 + aay^2 * + a^4 = 0$. In the ſubſtitution of $x = \frac{aa}{y}$, inſtead of the given quantity a, if we had taken any other, we ſhould have arrived at the ſame concluſion, but then the transformed equation would have involved fractions.

163. If,

163. If, in the propofed equation, not the fecond term, but the third, or fourth, &c. fhould be wanting, by the fame method we might make that term to vanifh, which is equally diftant from the laft term, as the abfent term is diftant from the firft. Or any other on a certain condition.

164. And on the contrary, if one or more terms be wanting in an equation, we may always make it compleat, by taking a new unknown quantity, *plus* or *minus* fome known quantity, and making it equal to the unknown quantity of the equation, and then the transformed equation will have all it's terms compleat. Moreover, if we would have the transformed equation to be of a fuperior degree, let every term of the propofed equation be multiplied by fuch a power of the unknown quantity, by which we would have the degree to be increafed, and then the fubftitution may be made. Thus, the equation $x^4 - a^4 = 0$ being given, if we would have it to be changed into another which is compleat, and of the fixth degree, let it be made $x^6 - a^4x^2 = 0$, and then making the fubftitution of $x = z \pm a$, (where by a is underftood any known quantity,) and we fhall have the equation required. The calculation, for brevity, is omitted. Or an equation may be completed or raifed higher.

165. When equations are reduced to fuch a form, as that they have their greateft term pofitive, and without a co-efficient except unity; that they may be free from fractions and furds, and compared to nothing, in order to judge whether the problem propofed be of that degree as is fhown by the equation, we muft examine whether it have a divifor of one, of two, or more dimenfions, by which, being divided, it may be reduced to a lower degree. For the problem is properly of that degree to which the equation may be reduced, and not of the degree of the firft equation. If a cubick equation have a divifor of one dimenfion, by being divided by that, it may be reduced to two dimenfions; and the two roots of this, (which will be had by the rules delivered at § 73, 74,) and the divifor, will be the three roots of the propofed equation. Whence the problem, which has brought us to fuch an equation, is not really cubical but plane, and may be conftructed by ruler and compaffes only, that is, by right lines and circles. If an equation of the fourth degree have two divifors of one dimenfion, and if it be divided by them, it will be reduced to two dimenfions; the roots of which, together with the two divifors, will be the four roots of the propofed equation, and therefore the problem will be plane. After the fame manner, if it have one divifor of two dimenfions, another of two dimenfions will be the quotient, the roots of which, together with the roots of the divifor, will be the four roots of the propofed equation, and therefore the problem is plane. Further, if it have one divifor only of one dimenfion, the reduced equation will be of three, and the problem will be folid indeed, but of the third degree only, and not of the fourth as it feemed to be. If an equation of the fifth degree fhall have three divifors of one dimenfion, or one of one and one of two, (which is the fame cafe as if it had two of two dimenfions, becaufe Problems are often reduced to a lower degree by divifion.

then it will neceſſarily have one of one dimenſion alſo,) it will be reduced to two dimenſions, and therefore the five roots may be had, and the problem will be plane. If it have only one of one dimenſion, it will be reduced to the fourth degree, and the problem will be of the ſame degree. If it ſhall have two of one dimenſion, or one of two, it will be reduced to the third degree, and the problem will be of the ſame. And the like of others. The manner of finding diviſors of one dimenſion has been taught before, at § 56.

And ſometimes by compound diviſors.

166. But beſides, as equations may have diviſors of two or more dimenſions, whether rational or irrational, we may operate with them in like manner, and, by a like way of reaſoning, we muſt attempt the diviſion of the propoſed equation; but, firſt, having tried the diviſion, by diviſors of one dimenſion, which ought always to precede, whatever the equation may be.

How equations of the fourth degree may often be reduced by two quadratick diviſors.

167. The manner of finding theſe diviſors for equations of the fourth degree may be this following. For thoſe of the third degree are either irreducible, or may be reduced by a rational and linear diviſor, being free from radicals, as is here ſuppoſed.

Admitting, then, that the equation of the fourth degree is not reducible by a diviſor of one dimenſion only; let the ſecond term be taken away (§ 160), and, for example's ſake, let there be produced this equation, $x^4 * - 17aax^2 - 20a^3x - 6a^4 = 0$. Let this be ſuppoſed equal to the product of theſe two equations of the ſecond degree, $x^2 + yx + z = 0$, and $x^2 - yx + u = 0$, in which y, z, u, are general quantities, which are to be determined afterwards as occaſion may require; and z and u may have any ſign. The product of theſe two equations will be

$$\begin{array}{llll} x^4 * & + zx^2 & - yzx & + uz = 0. \\ & - yyx^2 & + yux & \\ & + ux^2 & & \end{array}$$

Now let this equation be compared, term by term, with the equation propoſed, and, from the compariſon of the third terms in each, we ſhall have $z = -17a^2 + y^2 - u$. From the compariſon of the fourth terms, it will be $u = \frac{-20a^3}{y} + z$; and, inſtead of z, putting it's value already found, that we may have u expreſſed by y only, and known quantities, it will be $u = -\frac{20a^3}{2y} - \frac{17}{2}aa + \frac{1}{2}yy$. And, putting this value of u in the equation $z = -17aa + yy - u$, we ſhall have $z = -\frac{17}{2}aa + \frac{1}{2}yy + \frac{20a^3}{2y}$. From the compariſon of the laſt terms, we ſhall have $uz = -6a^4$, and, inſtead of z and u, putting their values expreſſed by y only, and known quantities, it will be $\frac{289}{4}a^4 - \frac{34}{4}aayy - \frac{400a^6}{4yy} + \frac{1}{4}y^4 = -6a^4$; or, reducing to a common denominator,

$y^6 -$

$$y^6 - 34a^2y^4 \begin{array}{l} + 289a^4y^2 \\ + 24a^4y^2 \end{array} - 400a^6 = 0.$$ This transformed equation may be considered as of the third degree, because it involves neither y^5, nor y^3, nor y. In this equation, let the divisors of the last term be found, and, because it may be considered as of the third degree, though it is really of the sixth, try if it be divisible by $yy \pm$ these divisors, among which we are to choose those only of two dimensions, as is plain. And it will be found divisible by $yy - 16aa = 0$, whence it will be $yy = 16aa$, and $y = \pm 4a$. This value of y being substituted in the equations $u = -\frac{20a^3}{2y} - \frac{17}{2}aa + \frac{1}{2}yy$, and $z = \frac{20a^3}{2y} - \frac{17}{2}aa + \frac{1}{2}yy$, we shall have $u = -3aa$, $z = 2aa$. Therefore the two subsidiary equations $x^2 + yx + z = 0$, and $x^2 - yx + u = 0$, must be $x^2 + 4ax + 2aa = 0$, and $x^2 + 4ax - 3aa = 0$, into which the equation $x^4 * - 17a^2x^2 - 20a^3x - 6a^4 = 0$ may be resolved, by dividing by either of them.

But the roots of these are (§ 74.) $x = -2a \pm \sqrt{2aa}$ for the first, and $x = 2a \pm \sqrt{7aa}$ for the second; which are therefore the roots of the given equation, being all four real, one positive and three negative.

If the transformed equation should not have any divisor, it would be to no purpose to seek another in this case; for neither would the proposed equation admit of any.

Although in the value of y we have $y = \pm 4a$, yet I have made use of the positive sign only, because it is indifferent whether we take the positive or the negative root, the result being the same in both cases. For, if we put $y = -4a$, it will be $u = 2aa$, $z = -3aa$, and the two equations will be the same as before, that is, $x^2 - 4ax - 3aa = 0$, and $x^2 + 4ax + 2aa = 0$.

Let the equation be $$x^4 - 2ax^3 \begin{array}{l} + 2a^2x^2 \\ - ccx^2 \end{array} - 2a^3x + a^4 = 0.$$ Taking away the second term, by the substitution of $x = z + \frac{1}{2}a$, it will be changed into $$z^4 * \begin{array}{l} + \frac{1}{2}a^2z^2 \\ - ccz^2 \end{array} \begin{array}{l} - a^3z \\ - ac^2z \end{array} \begin{array}{l} + \frac{5}{16}a^4 \\ - \frac{1}{4}a^2c^2 \end{array} = 0.$$ Wherefore, making a comparison of this with the equation $$z^4 * \begin{array}{l} + uz^2 \\ - y^2z^2 \\ + pz^2 \end{array} \begin{array}{l} - pyz \\ + uyz \end{array} + pu = 0,$$ which is the product of the two equations $z^2 + yz + p = 0$, and $z^2 - yz + u = 0$; from the comparison of the third terms, as usual, we shall have $p = yy - u + \frac{1}{2}aa - cc$. From the comparison of the fourth terms, we shall have $u = p \frac{-a^3 + acc}{y}$; or, instead of p, putting it's value, $u = \frac{1}{2}yy + \frac{1}{4}aa - \frac{1}{2}cc - \frac{a^3 + acc}{2y}$; and therefore $p = \frac{1}{2}yy + \frac{1}{4}aa - \frac{1}{2}cc + \frac{a^3 + acc}{y}$. Lastly, from the comparison of the

laſt terms, we ſhall have $pu = \frac{5}{16}a^4 - \frac{1}{4}aacc$; or, ſubſtituting the values of p and u, it will be $\left.\begin{matrix} y^6 + aay^4 - a^4y^2 - a^6 \\ - 2ccy^4 + c^4y^2 - 2a^4c^2 \\ - a^2c^4 \end{matrix}\right\} = 0.$

Now the diviſors of the laſt term, meaning thoſe of two dimenſions, are aa and $aa + cc$, and the diviſion will ſucceed by $yy - aa - cc = 0$. Therefore it will be $yy = aa + cc$, and $y = \pm\sqrt{aa + cc}$. Whence $u = \frac{3}{4}aa - \frac{a^3 + ac^2}{2\sqrt{aa + cc}}$, $p = \frac{3}{4}aa + \frac{a^3 + acc}{2\sqrt{aa + cc}}$; and the two equations $z^2 + yz + p = 0$, and $z^2 - yz + u = 0$, will be $zz + z\sqrt{aa + cc} + \frac{3}{4}aa + \frac{a^3 + ac^2}{2\sqrt{aa + cc}} = 0$, and $zz - z\sqrt{aa + cc} + \frac{3}{4}aa - \frac{a^3 + ac^2}{2\sqrt{aa + cc}} = 0$, or $zz + z\sqrt{aa + cc} + \frac{3}{4}aa + \frac{1}{2}a\sqrt{aa + cc} = 0$, and $zz - z\sqrt{aa + cc} + \frac{3}{4}aa - \frac{1}{2}a\sqrt{aa + cc} = 0$.

Theſe two equations, being reſolved, will give us four values of z; $z = -\frac{1}{2}\sqrt{aa + cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc - \frac{1}{2}a\sqrt{aa + cc}}$ from the firſt equation, and $z = \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc + \frac{1}{2}a\sqrt{aa + cc}}$ from the ſecond equation. And, becauſe theſe are the diviſors of the equation $\left.\begin{matrix} z^4 * + \frac{1}{2}a^2z^2 - a^3z + \frac{5}{16}a^4 \\ - ccz^2 - ac^2z - \frac{1}{4}a^2c^2 \end{matrix}\right\} = 0$, the ſame roots ſhall alſo belong to this equation. And now, making the ſubſtitution of $x = \frac{1}{2}a + z$, we ſhall have $x = \frac{1}{2}a - \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc - \frac{1}{2}a\sqrt{aa + cc}}$, and $x = \frac{1}{2}a + \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc + \frac{1}{2}a\sqrt{aa + cc}}$, which are the four roots or values of the propoſed equation.

This reduction may be performed by a general canon.

168. But a general formula or canon may be formed, as well for the tranſformed equation as for the two ſubſidiary equations, which are aſſumed in order to obtain the diviſors; to which formulas any equation whatever of the fourth degree, in which the ſecond term is wanting, or taken away, may be univerſally applied. Therefore let there be this general equation $x^4 * \pm px^2 \pm qx \pm r = 0$; and taking the two ſubſidiary equations, $x^2 + yx + z = 0$, and $x^2 - yx + u = 0$, and finding their product, $\begin{matrix} x^4 * + zx^2 - yzx + uz = 0, \\ - yyx^2 + uyx \\ + ux^2 \end{matrix}$

let it be compared, term by term, with the equation propoſed. Now, from the compariſon of the third terms, we ſhall have $z = \pm p + yy - u$. From the compariſon of the fourth, $u = z \pm \frac{q}{y}$; and, inſtead of z, it's value being

5 ſubſtituted,

ſubſtituted, it will be $u = \pm \frac{1}{2}p + \frac{1}{2}yy \pm \frac{q}{2y}$; where it is $+ p$, if in the propoſed equation the third term be poſitive, and $- p$, if negative. And thus alſo for q, if the fourth term be poſitive, and $- q$, if negative. And this being put inſtead of u in the firſt compariſon, we ſhall have $z = \pm \frac{1}{2}p + \frac{1}{2}yy \mp \frac{q}{2y}$; that is, $+ p$, if the third term of the propoſed equation be poſitive, and $- p$, if negative. And, on the contrary, $- q$, if the fourth term be poſitive, and $+ q$, if negative. From the compariſon of the laſt terms, we ſhall find $zu = \pm r$, that is, $\pm \frac{1}{2}p + \frac{1}{2}yy \pm \frac{q}{y}$ into $\pm \frac{1}{2}p + \frac{1}{2}yy \mp \frac{q}{2y} = \pm r$; and, by actual multiplication, and reducing to a common denominator, it will be $y^6 \pm 2py^4 \begin{array}{l} + p^2y^2 \\ \mp 4ry^2 \end{array} - qq = 0$, the transformed equation, which may be called cubick; in which it will be $+ 2p$, if the third term of the propoſed equation be poſitive, and $- p$, if negative. And it will be $- 4r$, if the laſt term of the propoſed equation be poſitive, but $+ 4r$, if negative. In the two ſubſidiary equations, inſtead of z and u, if we put their values found before, they will be $xx + yx \pm \frac{1}{2}p + \frac{1}{2}yy \mp \frac{q}{2y} = 0$, and $xx - yx \pm \frac{1}{2}p + \frac{1}{2}yy \pm \frac{q}{2y} = 0$. Wherefore, if the transformed equation ſhall be diviſible by $yy \pm$ a diviſor of two dimenſions of the laſt term, we ſhould have the value of y, which, being ſubſtituted in theſe two laſt equations, will ſupply us with diviſors of the propoſed equation. And if the transformed equation be not diviſible, neither will the propoſed be ſo.

Let the given equation be $x^4 * - 4a^2x^2 - 8a^3x + 35a^4 = 0$. Comparing this with the canonical equation, it will be $p = 4aa$, $q = 8a^3$, $r = 35a^4$; and therefore the transformed equation will be $y^6 - 8a^2y^4 \begin{array}{l} + 16a^4y^2 \\ - 140a^4y^2 \end{array} - 64a^6 = 0$, that is, $y^6 - 8a^2y^4 - 124a^4y^2 - 64a^6 = 0$. And the two ſubſidiary equations will be $x^2 + yx - 2aa + \frac{1}{2}yy + \frac{4a^3}{y} = 0$, and $x^2 - yx - 2aa + \frac{1}{2}yy - \frac{4a^3}{y} = 0$. Now, finding the diviſors of the laſt term, becauſe the tranſformed equation is diviſible by $yy - 16aa = 0$, we ſhall have $yy = 16aa$, and thence $y = 4a$; which values, being ſubſtituted in the two ſubſidiary equations, will give $x^2 + 4ax + 7aa = 0$, and $x^2 - 4ax + 5aa = 0$, which are the diviſors of the given equation; the four roots of which are $x = -2a \pm \sqrt{-3aa}$, and $x = 2a \pm \sqrt{-aa}$, all imaginary.

169. Some-

Sometimes a biquadratick may be reduced to a quadratick.

169. Sometimes it will be fufficient only to take away the fecond term of the equation, in order to reduce it to a plane, and fo to fpare any further operation. Thus, for example, it will be in the equation

$$x^4 + 2cx^3 - 2acx^2 - 2aacx - aacc = 0 \; \begin{matrix} \\ + ccx^2 \end{matrix}$$

which, becaufe it is not reducible by any divifor of the laft term, if we take away the fecond term by making $x = y - \frac{1}{2}c$, will be changed into this,

$$\left.\begin{matrix} y^4 * - 2a^2y^2 * + \frac{1}{16}c^4 \\ - \frac{1}{2}c^2y^2 \quad - \frac{1}{2}a^2c^2 \end{matrix}\right\} = 0;$$

an affected quadratick equation, the roots of which, being diminifhed by the quantity $\frac{1}{2}c$, by the fubftitution of $x = y - \frac{1}{2}c$, will be the fame as of the propofed equation.

Sometimes higher equations may be refolved by this method.

170. This method requires, that the fecond term fhould be taken away from the equation, nor can it be extended beyond equations of the fourth degree. But here is another method, which does not oblige us to take away any term, and which may be applied, not only to equations of the fourth degree, but to thofe of the fifth or fixth, and fometimes to thofe of ftill higher degrees. Let the given equation be

$$x^4 + ax^3 + a^2x^2 - a^2bx - a^3b = 0;$$
$$- abx^2$$

and let there be taken two fubfidiary equations of the fecond degree, $x^2 + yx + u = 0$, and $x^2 + sx + z = 0$, in which the indeterminates, y, u, s, z, are to be determined afterwards as occafion may require. The product of thefe will be

$$\begin{matrix} x^4 + yx^3 + ux^2 + usx + zu = 0, \\ + sx^3 + syx^2 + zyx \\ + zx^2 \end{matrix}$$

which is to be compared, term by term, with the propofed equation. From the comparifon of the fecond terms, we fhall have $s = a - y$; from the comparifon of the laft terms, $z = -\frac{a^3b}{u}$; and from the comparifon of the fourth, $yz + su = -a^2b$: and, inftead of s and z, fubftituting their values, that we may have an equation expreffed by y and u only, and known quantities, it will be $y = \frac{auu + aabu}{uu + a^3b}$. And, becaufe we have found $zu = -a^3b$, from the comparifon of the laft terms, therefore u ought to be a divifor of $-a^3b$. Whence let the divifors of $-a^3b$ of two dimenfions be found, (for thofe of one or three dimenfions will not ferve to be fubfidiary equations of the fecond degree,) which are $\pm ab$, $\pm aa$. Let us begin by taking, inftead of u, one of thefe divifors, for example ab, which, being fubftituted in the equation $y = \frac{auu + aabu}{uu + a^3b}$, gives $y = \frac{2ab}{a+b}$. Therefore, putting thefe values of y and u in the fubfidiary equation

$x^2 +$

$x^2 + yx + u = 0$, it will be $x^2 + \frac{2abx}{a+b} + ab = 0$. And by this, if we try the divifion of the propofed equation, and if it fhould fucceed, then $x^2 + \frac{2abx}{a+b} + ab = 0$ would be one divifor, and the quotient would be the other. But, becaufe the divifion does not fucceed, we muft make another trial, by taking, inftead of u, the other divifor $-ab$ of the laft term, and it will be $y = 0$; and therefore the fubfidiary equation $x^2 + yx + u = 0$ will become $x^2 - ab = 0$, by which the propofed equation being divided, it will fucceed by giving the quotient $x^2 + ax + aa = 0$. So that the divifors of the propofed equation are $xx - ab = 0$, and $xx + ax + aa = 0$.

Alfo, inftead of u, taking the divifor aa of the laft term, by which we fhall find $y = a$, and the fubfidiary equation will be $xx + ax + aa = 0$. The divifion by this will fucceed, giving the quotient $xx - ab = 0$; that is, the very fame divifors as before.

When all the divifors of the laft term are put in the place of u, and if the operation will not fucceed by any, it may then be concluded, that the equation propofed cannot be depreffed, at leaft by this method, and that the Problem remains of fuch a degree as the equation indicates.

But, without trying the divifion, taking, inftead of u, every one of the divifors of two dimenfions of the laft term, and the correfpondent values of y, s, z, we may fubftitute them in their ftead in the fubfidiary formulas, $xx + yx + u = 0$, and $xx + sx + z = 0$. And if the product of thefe will give the propofed equation, they will be the divifors required. Thus, taking, inftead of u, the divifor $-ab$, we fhall have $y = 0$, and therefore $s = a$, $z = aa$, and the two fubfidiary equations will be $xx - ab = 0$, and $xx + ax + aa = 0$, the product of which will give us the propofed equation.

Let the equation $\begin{array}{l} x^4 - 2ax^3 + 2aax^2 - 2a^3x + a^4 \\ \qquad\qquad\quad\; - ccx^2 \end{array}$ be given, and let it be compared with the product of the two fubfidiary equations

$$\begin{array}{llll} x^4 + yx^3 & + ux^2 & + sux & + zu = 0. \\ \quad + sx^3 & + syx^2 & + zyx & \\ & + zx^2 & & \end{array}$$

From the comparifon of the fecond terms, we fhall have $s = -2a - y$. From the comparifon of the laft terms, $z = \frac{a^4}{u}$. We muft take the comparifon of the third, and not of the fourth, in order to have the value of y expreffed by c, (which letter muft neceffarily be in the divifor, which could not be had from the comparifon of the fourth,) it will be then $u + sy + z = 2aa - cc$. And fubftituting the values of s and z, it will be $yy + 2ay = \frac{a^4}{u} - 2aa + cc + u$; in which fubftituting, inftead of u, one of the divifors $\pm aa$ of the laft term, fuppofe $+ aa$, and refolving the

the equation, it will be $y = -a \pm \sqrt{aa+cc}$. And putting, in the equation $xx + yx + u = 0$, the values of u and y, (taking for the fign of the radical quantity either *plus* or *minus* as we pleafe, becaufe it will be all the fame at laft,) we fhall have $xx - ax + x\sqrt{aa+cc} + aa = 0$, by which the divifion of the propofed equation will fucceed, making the quotient $xx - ax - x\sqrt{aa+cc} + aa = 0$; and confequently the four roots of the propofed equation will be $x = \frac{1}{2}a - \frac{1}{2}\sqrt{aa+cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc - \frac{1}{2}a\sqrt{aa+cc}}$, and $x = \frac{1}{2}a + \frac{1}{2}\sqrt{aa+cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc + \frac{1}{2}a\sqrt{aa+cc}}$.

Let the equation be $x^4 + 2bx^3 + bbx^2 - a^3b = 0$, and let it be compared with the product of the two fubfidiary formulas as before. From the comparifon of the fecond terms, we fhall have $s = 2b - y$. From the comparifon of the laft, $z = -\frac{a^3b}{u}$. From the comparifon of the fourth, $zy + su = 0$; and fubftituting the values of s and z, it will be $-\frac{a^3by}{u} + 2bu - uy = 0$, that is, $y = \frac{2buu}{a^3b + uu}$. But, taking every one of the rational divifors, $\pm aa$, $\pm ab$, of the laft term, and fubftituting in the place of u, and doing the reft as ufual, the operation does not fucceed. Therefore we muft try by means of the irrational divifors $\pm a\sqrt{ab}$ of the laft term; and therefore putting, inftead of u, the irrational divifor $a\sqrt{ab}$, it will be $y = b$. Wherefore the fubfidiary equation $xx + yx + u = 0$ will become $xx + bx + a\sqrt{ab} = 0$, by which the propofed equation being divided, there will arife the quotient $xx + bx - a\sqrt{ab} = 0$.

Exemplified in equations of the fifth degree.

171. As to equations of the fifth degree, it is manifeft, that if they be not divifible by a linear divifor, as already fuppofed, they cannot be divided but by one of the fecond degree, and one of the third. Therefore for fuch equations muft be taken two fubfidiary equations, one of the third degree, and another of the fecond, and the product of thefe muft be compared, term with term, with the propofed equation, in like manner as before.

Let this therefore be the given equation, $x^5 - 4ax^4 + 6aax^3 - 8a^3x^2 + 5a^4x - a^5 = 0$. And let us take the two fubfidiary equations $xx + yx + u = 0$, and $x^3 + tx^2 + sx + z = 0$. Of thefe the product is

$$\begin{array}{llllll} x^5 & + yx^4 & + ux^3 & + tux^2 & + sux & + zu = 0 \\ & + tx^4 & + tyx^3 & + syx^2 & + zyx & \\ & & + sx^3 & + zx^2 & & \end{array}$$

which is to be compared with the propofed equation. Now, from the comparifon of the fecond terms, we fhall have $t = -4a - y$. From the comparifon of the laft terms, $z = -\frac{a^5}{u}$. From the comparifon of the fifth, $s = \frac{5a^4 - zy}{u}$; or the value of z being

being fubftituted, $s = \frac{5a^4}{u} + \frac{a^5y}{uu}$. From the comparifon of the third, we fhall have finally $u + ty + s = 6aa$; and, inftead of t and s, putting their values, in order to obtain an equation expreffed by y, u, and known quantities only, it will be $yy + 4ay - \frac{a^5y}{uu} = -6aa + u + \frac{5a^4}{u}$. And becaufe, from the comparifon of the laft terms, we have $z = -\frac{a^5}{u}$, therefore u will be a divifor of $-a^5$. So that, finding all the divifors of two dimenfions of $-a^5$, they are to be fubftituted, one by one, in the foregoing equation, in order to have the value of y, which is then to be put inftead of y in the fubfidiary equation $xx + yx + u = 0$, as alfo the value of u. And if the divifion of the given equation fhall fucceed by this, we fhall have our defire. Now the divifors of two dimenfions of the laft term are $\pm aa$. Let us take $+ aa$, which being fubftituted inftead of u in the equation aforegoing, we fhall have $yy + 3ay = 0$, that is, $y = 0$, and $y = -3a$. If, in the fubfidiary equation $xx + yx + u = 0$, we put the divifor $+ aa$ inftead of u, and befides, if we put 0, which is one of the values found, inftead of y, it will become $xx + aa = 0$, by which the divifion of the propofed equation does not fucceed. Therefore, inftead of y, we may put it's other value $-3a$, and we fhall have $xx - 3ax + aa = 0$, by which the divifion fucceeds, and gives $x^3 - ax^2 + 2aax - a^3 = 0$ in the quotient. If the operation had not fucceeded by means of the divifor $+ aa$, we muft have tried the divifor $- aa$; and if neither by this we had obtained our defire, we muft have concluded the equation to be irreducible, at leaft by this method.

Let the equation be $x^5 + ax^4 * \begin{matrix} + a^3x^2 \\ - a^2bx^2 \end{matrix} - aabbx - a^4b = 0$, which is to be compared, term by term, with the product of the two ufual fubfidiary equations; and from the comparifon of the fecond terms, we fhall find $t = a - y$. From the comparifon of the laft terms, $z = -\frac{a^4b}{u}$. From the comparifon of the fifth, $su + zy = -aabb$. Now, inftead of z, fubftituting it's value, it will be $s = -\frac{aabb}{u} + \frac{a^4by}{uu}$. From the comparifon of the third, we fhall have $u + ty + s = 0$, in which, inftead of t and s, putting their values, it will be $yy - ay - \frac{a^4by}{uu} = \frac{uu - aabb}{u}$. The divifors of two dimenfions of a^4b are $\pm aa$, and $\pm ab$. We muft try the operation by means of the divifor $- ab$. And therefore, inftead of u, putting it's value $- ab$ in the laft equation, it will be $yy - ay - \frac{aay}{b} = 0$. Thence $y = 0$, and $y = \frac{ab + aa}{b}$. In the fubfidiary equation $xx + yx + u = 0$, inftead of y let it's value $\frac{aa + ab}{b}$ be

be substituted, and $-ab$ instead of u, and it will be $xx + ax + \frac{aa}{b}x - ab = 0$, by which the division does not succeed. Therefore take the other value of y, which is o, and the subsidiary equation will be $xx - ab = 0$, by which the division of the proposed equation will succeed, and the quotient will be $x^3 + ax^2 + abx + a^3 = 0$.

We were at liberty to make a comparison between the fourth terms; but, for greater simplicity, I made choice of the third terms.

Equations of the sixth degree resolved.

172. Equations of the sixth degree, supposed not to be reducible by any linear divisor, cannot be otherwise reducible but either by three divisors of two dimensions, or by one of two dimensions and one of four, or by two of three dimensions. But it will be sufficient to examine the two cases, in which they are reducible by two of three dimensions, or by one of two and one of four. For as much as reducing them by one of two, the reduced equation will be of four dimensions, which may afterwards be reduced by two divisors of two dimensions, if the proposed equation be reducible by three of two dimensions.

Let the equation given be this: $x^6 - 13ax^5 + 45aax^4 - 71a^3x^3 + 57a^4x^2 - 16a^5x + 2a^6 = 0$, which is required to be reduced by one of two dimensions, and one of four. Let therefore be taken the two subsidiary equations $xx + yx + u = 0$, and $x^4 + px^3 + tx^2 + sx + z = 0$, of which the product is

$$\begin{array}{llllll} x^6 + px^5 & + tx^4 & + sx^3 & + zx^2 & + zyx & + zu = 0. \\ + yx^5 & + pyx^4 & + tyx^3 & + syx^2 & + sux & \\ & + ux^4 & + pux^3 & + tux^2 & & \end{array}$$

Now, from the comparison of the second terms, we shall have $p = -13a - y$. From the comparison of the last terms, $z = \frac{2a^6}{u}$. From the comparison of the third, $t + py + u = 45aa$; and by substituting the value of p, it will be $t = 45aa + 13ay + yy - u$. From the comparison of the sixth, $zy + su = -16a^5$; and putting here the value of z, it will be $s = -\frac{2a^6y}{uu} - \frac{16a^5}{u}$. From the comparison of the fifth, $z + sy + tu = 57a^4$; and substituting the values of z, s, and t, that we may have an equation expressed by u and y alone, and by the known quantities of the proposed equation, it will be at last $\frac{2a^6}{u} - \frac{2a^6y^2}{uu} - \frac{16a^5y}{u} + 45a^2u + 13ayu + uy^2 - u^2 = 57a^4$. That is, $yy + \frac{13au^3y - 16a^5uy + 2a^6u - 57a^4u^2 + 45a^2u^3 - u^4}{u^3 - 2a^6} = 0$. And, because the divisors of two dimensions of the last term $2a^6$ are $\pm aa$, and $\pm 2aa$, we must make a trial, by putting in this last equation, instead of u, the divisor $+aa$, and it will be $yy + 3ay + 11aa = 0$, which, being resolved, will give $y =$

$y = \frac{-3a \pm \sqrt{-35aa}}{2}$. Whence the fubfidiary formula $xx + yx + u = 0$ will be $xx - \frac{3a + \sqrt{-35aa}}{2}x + aa = 0$. But by this, even though we fhould take the alternative of the figns of the radical, the propofed equation is not divifible; nor will it fucceed if we fhould take the divifor $-aa$; therefore we muft take $+2aa$, and we fhall have $yy + 12ay + 20aa = 0$, that is, $y = -6a \pm 4a$, or $y = -10a$, and $y = -2a$. Take $y = -10a$, and fubftitute it in the fubfidiary formula $xx + yx + u = 0$, and $-10a$ inftead of y, and $+2aa$ inftead of u, and it will be $xx - 10ax + 2aa = 0$. But by this the divifion of the propofed equation does not fucceed. Therefore take the other value of y, or $-2a$, and the formula will be $xx - 2ax + 2aa = 0$, by which the divifion fucceeds, making in the quotient $x^4 - 11ax^3 + 21aax^2 - 7a^3x + a^4 = 0$.

Here it may not be amifs to obferve, that, inftead of the comparifon of the fifth terms, if I had made a comparifon of the fourth, I fhould have fallen upon the cubick equation $2y^3 + 26ay^2 + 81aay + 74a^3 = 0$. But the comparifon of the fifth terms has brought me to a quadratick equation only. Hence it may be feen, that the choice of the comparifon of fome terms rather than of others may be of good advantage. Yet, however, this cubick equation might have been of ufe; for, finding it's roots, which are $y + 2a = 0$, and $y + \frac{11a}{2} \pm \sqrt{47aa} = 0$, one of thefe, $y = -2a$, would have given me the fame equation $xx - 2ax + 2aa = 0$, by which the propofed equation may be divided.

Let $x^6 + 3ax^5 + 4aax^4 + 6a^3x^3 + 6a^4x^2 + 3a^5x + 2a^6 = 0$, be the given equation of the fixth degree, not reducible by a divifor of two dimenfions. Let us therefore attempt the reduction by two equations of three dimenfions, and let us take thefe two fubfidiary equations, $x^3 + yx^2 + px + u = 0$, and $x^3 + tx^2 + sx + z = 0$, of which this is the product;

$$\begin{array}{llllllll} x^6 & + yx^5 & + px^4 & + ux^3 & + tux^2 & + sux & + zu & = 0. \\ & + tx^5 & + tyx^4 & + ptx^3 & + psx^2 & + pzx & & \\ & & + sx^4 & + syx^3 & + zyx^2 & & & \\ & & & + zx^3 & & & & \end{array}$$

Now, from the comparifon of the fecond terms, we fhall have $t = 3a - y$. From the comparifon of the laft terms, $z = \frac{2a^6}{u}$. From the comparifon of the fixth, $su + pz = 3a^5$; and fubftituting the value of z, it will be $s = \frac{3a^5}{u} - \frac{2a^6p}{uu}$. From the comparifon of the third, $p + ty + s = 4aa$; and fubftituting the values of t and s, it will be $p = \frac{4aauu - 3a^5u + uuyy - 3auuy}{uu - 2a^6}$. From the

 comparifon

comparison of the fourth, $u + pt + sy + z = 6a^3$; and, inftead of t, s, z, fubftituting their values, that we may have another value of p, expreffed by u, y, and the known quantities of the equation, it will be $p = \frac{6a^3uu - u^3 - 3a^5uy - 2a^6u}{3auu - uuy - 2a^6y}$. Now, between thefe two values of p let an equation be made, to obtain the value of y expreffed by u only, and the given quantities of the equation. This will be $\frac{4aauu - 3a^5u - 3auuy + uuyy}{uu - 2a^6} = \frac{6a^3uu - u^3 - 3a^5uy - 2a^6u}{3auu - uuy - 2a^6y}$. Then, reducing to a common denominator, and ordering the equation by y, it will be

$$\frac{\begin{array}{rrr} y^3 - 6a^7uy^2 & + 8a^3uy & - 6a^3u^3 \\ - 6au^3y^2 & - 6a^5uuy & + 9a^6u^2 \\ & + 13a^2u^3y & - 12a^9u \\ & & + 4a^{12} \\ & & - u^4 \end{array}}{u^3 + 2a^6u} = 0.$$

And, becaufe it is $uz = 2a^6$, we fhall have u a divifor of $2a^6$. But the divifors of three dimenfions of $2a^6$ are $\pm a^3$, and $\pm 2a^3$. Whence, taking one of thefe inftead of u, fuppofe $+ a^3$, and fubftituting it in the laft equation, we fhall have $y^3 - 4ay^2 + 5aay - 2a^3 = 0$. From hence muft be extracted the values of y, one of which is $y = 2a$, which, being fubftituted in one of the values of p inftead of y, and putting inftead of u the divifor a^3, it will be $p = aa$. Wherefore, fubftituting thefe values of y, p, and u, in the fubfidiary formula $x^3 + yx^2 + px + u = 0$, it will become $x^3 + 2ax^2 + aax + a^3 = 0$, by which the propofed equation being divided, will give the quotient $x^3 + ax^2 + aax + 2a^3 = 0$. If the divifion had not fucceeded by taking $y = 2a$, I muft have taken $y = a$. And if I had not attained my purpofe by this, I muft have made trials with every one of the other divifors, repeating the fame operations. And if it had fucceeded by none of thefe, the propofed equation could not have been depreffed, at leaft not by this method, but would have remained of the fixth degree.

Let $x^6 + ax^5 + aax^4 + 3a^3x^3 + a^4x^2 + a^5x + 2a^6 = 0$ be the equation, which is to be compared with the product of the two fubfidiary equations, as in the foregoing example. From the comparifon of the fecond terms, we fhall have $t = a - y$. From the comparifon of the laft terms, $z = \frac{2a^6}{u}$. From the comparifon of the fixth, $su + pz = a^5$; and, inftead of z, putting it's value, it will be $s = \frac{a^5}{u} - \frac{2a^6p}{uu}$. From the comparifon of the third, $p + ty + s = aa$; and putting the values of t and s, it is $p = \frac{aauu - auuy + uuyy - a^5u}{uu - 2a^6}$. From the comparifon of the fourth, $u + pt + sy + z = 3a^3$; and fubftituting the

the values of z, s, t, in order to have another value of p, expreffed only by u, y, and known quantities, it will be $p = \frac{3a^3uu - a^3uy - 2a^6u - u^3}{auu - uuy - 2a^6y}$. Make an equation between thefe two values of p, that we may have the value of y given by u only and known quantities; and when all the neceffary operations are performed, it will be

$$\left.\begin{array}{l} y^3 - 2au^3y^2 + 2aau^3y + 2a^3u^3 \\ \quad - 2a^7uy^2 - 2a^5uuy + a^6u^2 \\ \qquad + 2a^8uy - 6a^9u \\ \qquad\qquad - u^4 \\ \qquad\qquad + 4a^{12} \\ \hline u^3 + 2a^6u \end{array}\right\} = 0.$$

The divifors of three dimenfions of $2a^6$ are $\pm a^3$, and $\pm 2a^3$. Inftead of u, take the divifor $+ a^3$, to be fubftituted in this laft equation, which then will be reduced to $y^3 - \frac{4}{3}ay^2 + \frac{2}{3}aay = 0$. And dividing by y, it will be $y = 0$, and $y^2 - \frac{4}{3}ay + \frac{2}{3}aa = 0$; that is, $y = \frac{2a \pm \sqrt{-2aa}}{3}$. Of thefe three values of y take the firft, or $y = 0$, and fubftitute this inftead of y in one of the two values of p, and a^3 inftead of u, and it will be $p = 0$. Then the fubfidiary equation $x^3 + yx^2 + px + u = 0$ will become $x^3 + a^3 = 0$; by which the propofed equation being divided, will give $x^3 + ax^2 + a^2x + 2a^3 = 0$ for the quotient.

In fuch equations as thefe, if it were known at firft that they are divifible by a divifor, in which fome term is wanting, much labour might be fpared, by taking one of the two fubfidiary equations without that term. But, becaufe this is not known, we may firft try the operation with one of thofe fubfidiary equations, which wants either one or more terms. Neverthelefs, becaufe the labour would be loft, if the propofed equation be not reducible by this means, and there will be need at laft, notwithftanding this compendium, to have recourfe to compleat fubfidiary equations, it will be better at once to ufe this general method, becaufe it gives the divifors in both cafes.

Without repeating the operations at every example, I might have formed a general canon, to which every particular equation might be referred, after the fame manner as that at § 168. But befides, as this may create fome confufion, it feems to me that actual operations made on purpofe afford more light, and have a better effect; therefore I have rather chofe to confine myfelf to them.

173. After the fame analogy, we may apply this method to equations of a fuperior order, but the calculation increafes beyond meafure. For, if we are to

Applied to higher equations.

to reduce an equation of the eighth degree, for example, by means of two equations of the fourth, in which no term is wanting, each of the two subsidiary equations muſt have four indeterminates, or general co-efficients. Whence, if we conſider one of theſe equations, ſuch as this, $x^4 + yx^3 + px^2 + qx + u = 0$, and take for u one of the diviſors of the laſt term of the propoſed equation, there will remain three indeterminates, y, p, q, to be determined by the uſual compariſons, in which there will occur ſolid equations, whoſe roots are to be extracted, in order that the operation may proceed.

PROBLEM I.

Applied to the ſolution of an arithmetical problem.

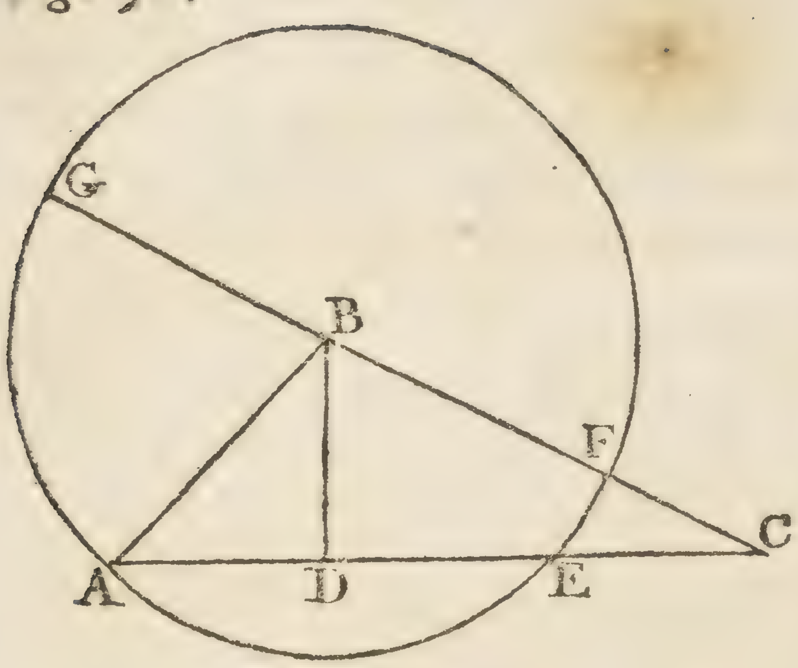
174. To find four numbers, which exceed one another by unity, and their product is 100.

Make the firſt number equal to x, the ſecond will be $x + 1$, the third $x + 2$, and the fourth $x + 3$. Therefore their product will be $x^4 + 6x^3 + 11x^2 + 6x = 100$, or $x^4 + 6x^3 + 11x^2 + 6x - 100 = 0$. Now, becauſe this equation is not diviſible by any diviſor of the laſt term, we muſt make the ſecond term to vaniſh by the ſubſtitution of $x = z - \frac{3}{2}$, and there will ariſe the equation $z^4 * - \frac{5}{2}z^2 * - \frac{1591}{16} = 0$, which is an affected quadratick, the roots of which are $zz = \frac{5}{4} \pm \sqrt{101}$, and therefore $z = \pm\sqrt{\frac{5}{4} \pm \sqrt{101}}$. Whence we ſhall have $x = -\frac{3}{2} \pm \sqrt{\frac{5}{4} \pm \sqrt{101}}$. Therefore, of the four values of x, two are real, that is, $x = -\frac{3}{2} \pm \sqrt{\frac{5}{4} + \sqrt{101}}$, and the other two are imaginary. If we take one of the real roots, $-\frac{3}{2} + \sqrt{\frac{5}{4} + \sqrt{101}}$, for the firſt number of the four that are required, then $-\frac{1}{2} + \sqrt{\frac{5}{4} + \sqrt{101}}$ will be the ſecond, $\frac{1}{2} + \sqrt{\frac{5}{4} + \sqrt{101}}$ will be the third, and $\frac{3}{2} + \sqrt{\frac{5}{4} + \sqrt{101}}$ will be the fourth: the product of which numbers will be found to be 100. If we ſhould take the other real value of x, that is, $-\frac{3}{2} - \sqrt{\frac{5}{4} + \sqrt{101}}$, for the firſt number, then $-\frac{1}{2} - \sqrt{\frac{5}{4} + \sqrt{101}}$ would be the ſecond, $\frac{1}{2} - \sqrt{\frac{5}{4} + \sqrt{101}}$ would be the third, and $\frac{3}{2} - \sqrt{\frac{5}{4} + \sqrt{101}}$ would be the fourth; the product of which numbers would alſo be found to be 100.

PRO-

PROBLEM II.

Fig. 91.



175. In the right-angled triangle ABC the leffer fide AB is given, and, letting fall the perpendicular BD to the bafe AC, the difference of the fegments AD, DC, of the fame bafe AC is given alfo; it is required to find FC, the difference of the fides AB, BC. A geometrical problem.

With centre B, diftance BA, let the circle AEFG be defcribed, and make AB $= a$, CE $= b$, the given difference of the fegments AD, DC; and make FC, the difference required, $= x$. It will be GC $= 2a + x$, and, by the property of the circle, GC $\times$ CF $=$ AC $\times$ CE, that is, $2ax + xx =$ AC $\times b$, and therefore AC $= \frac{2ax + xx}{b}$. But, becaufe the angle ABC is a right angle, we fhall have the equation $\frac{4aaxx + 4ax^3 + x^4}{bb} = 2aa + 2ax + xx$, or, by reduction, $\begin{matrix} x^4 + 4ax^3 + 4aaxx - 2abbx - 2aabb = 0. \\ \quad\quad\quad\quad\quad - bbxx \end{matrix}$ Now this is not divifible by any divifor of the laft term, and therefore we muft take away the fecond term by the fubftitution of $x = z - a$; whence we fhall have the affected quadratick

$$\left.\begin{matrix} z^4 - 2aazz + a^4 \\ \quad - bbzz - aabb \end{matrix}\right\} = 0,$$

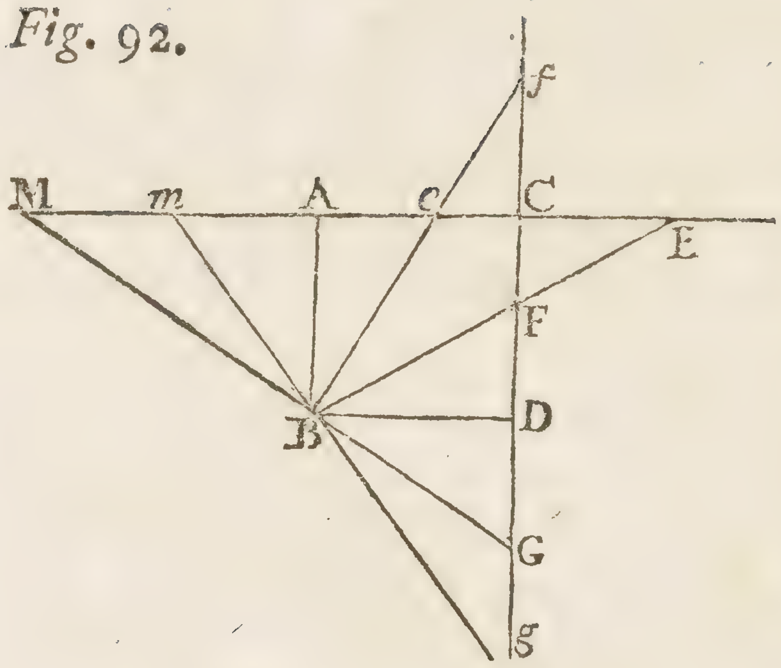
the roots of which are $zz = \frac{2aa + bb \pm \sqrt{8aabb + b^4}}{2}$, and thence $z = \pm\sqrt{\frac{2aa + bb \pm \sqrt{8aabb + b^4}}{2}}$. So that $x = -a \pm \sqrt{\frac{2aa + bb \pm \sqrt{8aabb + b^4}}{2}}$, which are the four roots, and all real, when a is greater than b. The root $x = -a + \sqrt{aa + \frac{1}{2}bb + b\sqrt{2aa + \frac{1}{4}bb}}$, which is pofitive, is adapted to the propofed Problem. The negative root $x = -a + \sqrt{aa + \frac{1}{2}bb - b\sqrt{2aa + \frac{1}{4}bb}}$ is adapted to the cafe, when the fide BC is lefs than the fide AB; the other two roots ferve for the angle ABG.

PRO-

PROBLEM III.

Another geometrical problem.

Fig. 92.



176. Having given the ſquare AD, in the ſide AC produced, to find ſuch a point E, that, drawing the right line EB to the angle B, the intercepted line EF may be equal to a given right line c.

Make BD $= a$, DF $= x$; it will be CF $= a - x$. And, drawing BFE, make FE $= c$. Now, by ſimilar triangles, ECF, BDF, it will be CF $(a - x)$. FE (c) :: FD (x) . FB $= \frac{cx}{a-x}$. But, becauſe of the right angle at D, it will be alſo FB $= \sqrt{aa + xx}$; whence we ſhall have the equation $\sqrt{aa + xx} = \frac{cx}{a-x}$; and, by ſquaring, $\frac{ccxx}{aa - 2ax + xx} = aa + xx$; and, reducing to a common denominator, and then ordering the equation, it is $x^4 - 2ax^3 \begin{smallmatrix} + 2aaxx \\ - ccxx \end{smallmatrix} - 2a^3x + a^4 = 0$, the roots of which may be ſeen, at § 167, 170, to be $x = \frac{1}{2}a - \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{\frac{1}{4}cc - \frac{1}{2}aa - \frac{1}{2}a\sqrt{aa + cc}}$, and $x = \frac{1}{2}a + \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{\frac{1}{4}cc - \frac{1}{2}aa + \frac{1}{2}a\sqrt{aa + cc}}$.

The two laſt roots are always real and poſitive; the latter of which, being leſs than a, determines the point F, through which the line BE being drawn, EF will be equal to the given line c, and reſolves the Problem propoſed. The other of the two, which is greater than a, determines the point f, to which drawing the right line Bf, gives us alſo cf equal to the given line, and ſerves as if the Problem had been propoſed by the angle ACf.

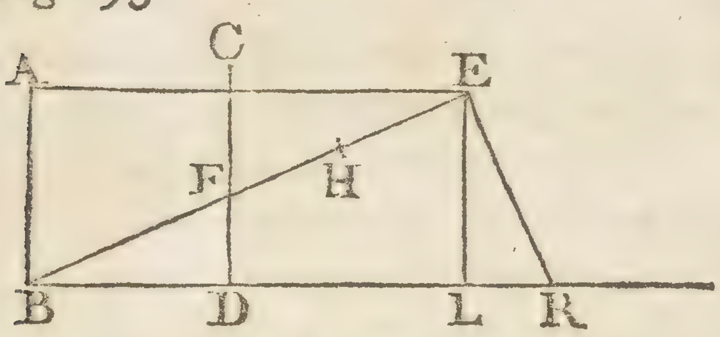
The two firſt roots are imaginary whenever cc is leſs than $8aa$, and the Problem will be impoſſible. But, ſuppoſing cc not leſs than $8aa$, the two roots are real and negative. Taking, therefore, DG $= \frac{1}{2}a - \frac{1}{2}\sqrt{aa + cc} + \sqrt{\frac{1}{4}cc - \frac{1}{2}aa - \frac{1}{2}a\sqrt{aa + cc}}$, and D$g$ $= \frac{1}{2}a - \frac{1}{2}\sqrt{aa + cc} - \sqrt{\frac{1}{4}cc - \frac{1}{2}aa - \frac{1}{2}a\sqrt{aa + cc}}$, and through the point B drawing the right lines GM, gm, they will both be equal to the given line c, and would ſerve were the Problem propoſed for the angle ACD.

177. Very

177. Very often, when the Problem is not really solid, but plane, it may appear as an equation of three dimensions, by making use of some certain line for the unknown quantity; but, by using some other line for the unknown quantity, it may put on the form of an equation of two dimensions only. I shall take an example of this in the foregoing Problem, in which, making $DF = x$, there has been found an equation of the fourth degree, by which means we have been obliged to take the trouble of reducing it. But, supposing E to be the point required, draw ER perpendicular to BE, which may meet BD produced in R, and EL perpendicular to BR. Then make $DR = x$, and, as before, $BD = a$, $FE = c$, and $BF = y$, another unknown quantity to be eliminated afterwards; it will be $BR = a + x$, $BE = c + y$. Now, because of similar triangles, BDF, ELR, it will be $ER = y$, because of $EL = CD = BD$. And, because of similar triangles, BRE, ERL, it will be BR . BE :: ER . EL. Therefore it will be $a + x \,.\, c + y :: y \,.\, a$; whence $cy + yy = aa + ax$. But, because of the right angle BER, the square of BR is equal to the sum of the squares of BE and ER; that is, $aa + 2ax + xx = 2yy + 2cy + cc$. Therefore, instead of $cy + yy$, putting it's value $aa + ax$, the equation will be $aa + 2ax + xx = 2aa + 2ax + cc$, that is, $x = \pm\sqrt{aa + cc}$.

How higher equations may sometimes be avoided.

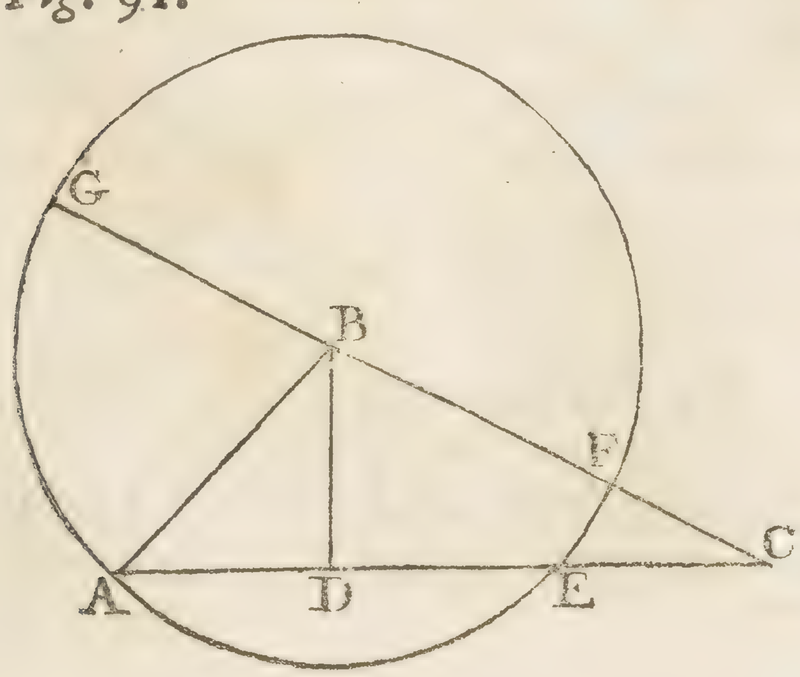
Fig. 93.



Again, after another manner. Bisect FE in H, and making $CD = a$, let the given line be $2c$, to which FE ought to be equal. And making $BH = x$, it will be $BF = x - c$, and $BE = x + c$. But $BEq - ABq = AEq$; therefore it will be $AE = \sqrt{xx + 2cx + cc - aa}$. Now, because of the similar triangles, BDF, BEA, it will be BF $(x - c)$. BD (a) :: BE $(x + c)$. AE $= \sqrt{xx + 2cx + cc - aa}$. Whence $ax + ac = \overline{x - c} \times \sqrt{xx + 2cx + cc - aa}$; and, by squaring and ordering the equation, it will be finally

$$\left.\begin{matrix} x^4 - 2aaxx - 2aacc \\ - 2ccxx + c^4 \end{matrix}\right\} = 0,$$ an affected quadratick equation, of which the four roots are $x = \pm\sqrt{aa + cc \pm a\sqrt{aa + 4cc}}$.

Fig. 91.



After the same manner in Prob. II. § 175, if, instead of making $FC = x$, I had denominated $BC = x$; by pursuing the same argumentation, I should have found the equation $$\left.\begin{matrix} x^4 - 2aaxx + a^4 \\ - bbxx - aabb \end{matrix}\right\} = 0,$$ an affected quadratick, of which the roots are $x = \pm\sqrt{aa + \frac{1}{2}bb \pm b\sqrt{2aa + \frac{1}{4}bb}}$, which agree with those before found.

Again,

Again, in a fimpler manner. Make AE $= x$, and, arguing as before, we fhould have the equation $xx + bx = 2aa$, and therefore $x = -\frac{1}{2}b \pm \sqrt{2aa+\frac{1}{4}bb}$. And, becaufe we fhould find the expreffion $-a + \sqrt{bb + 2bx + xx - aa}$ for FC, inftead of x putting the value now found, we fhould have what is required, or the fame value for FC as before.

Or otherwife, by finding two values of the fame quantity.

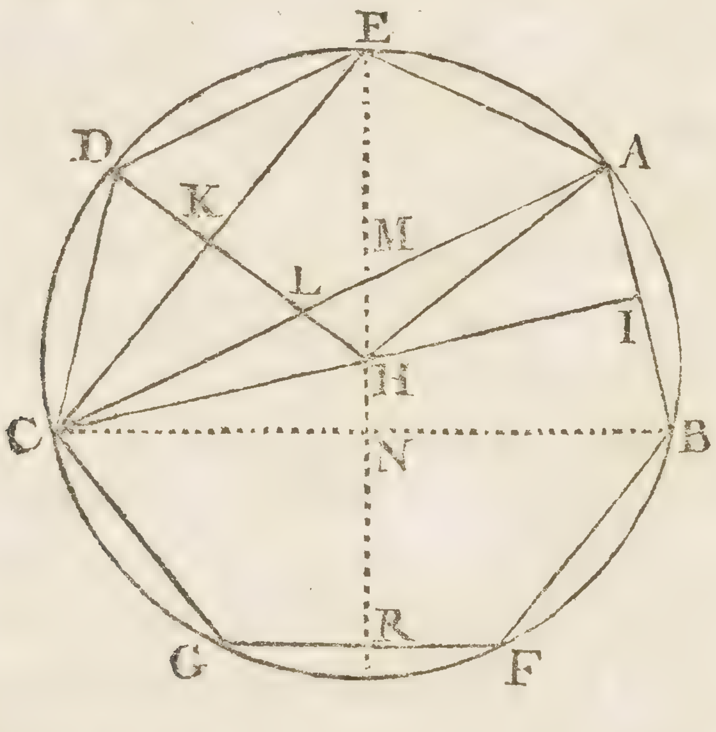
178. Another artifice may be tried for fuch like Problems, when they bring us to a folid equation, and yet are not fuch in their own nature. This is, retaining the fame line for the unknown quantity, by which the firft equation is found; then, by means of another property, to find a fecond equation, and to equal one to the other. From their comparifon, a third equation will arife of an inferior degree. See an example of this in the following Problem.

PROBLEM.

This exemplified in a geometrical problem.

179. In a given circle, to infcribe a regular heptagon.

Fig. 94.



Let the given circle be ABFGCDE, with centre H, radius HA $= r$, and let the fide of the heptagon be AB $=$ BF $=$ FG, &c. $= x$. Let AB be bifected in I; it will be AI $= \frac{1}{2}x =$ IB. And drawing IC, which will neceffarily pafs through the centre H, it will be HI $= \sqrt{rr - \frac{1}{4}xx}$, CI $= r + \sqrt{rr - \frac{1}{4}xx}$, CB $= \sqrt{2rr + 2r\sqrt{rr - \frac{1}{4}xx}}$. Let there be drawn CE and HD; the triangles CDK, HIA, will be fimilar, becaufe of the two right angles CKD, HIA, and of the angles DCK, AHI, the firft of which, becaufe it infifts on the arch DE, will be double to the angle ACI, which infifts on the half of DE, and therefore is equal to the angle AHI the double of the fame angle ACI. Hence we fhall have, by the fimilitude of thefe triangles, CK $= \frac{x}{r}\sqrt{rr - \frac{1}{4}xx} = \frac{\sqrt{4rrxx - x^4}}{2r}$, CE $= \frac{\sqrt{4rrxx - x^4}}{r}$, and HK $= \sqrt{rr - \frac{4rrxx - x^4}{4rr}} = \frac{2rr - xx}{2r}$. But the triangles CEN, CHK, are alfo fimilar, the two angles at K, N, being right ones, and the two angles KCH, CEN, are equal, becaufe they infift on two equal fegments. Therefore

 it

it will be $CN = \frac{\overline{2rr - xx} \times \sqrt{4rrxx - x^4}}{2r^3}$, and $CB = \frac{\overline{2rr - xx} \times \sqrt{4rrxx - x^4}}{r^3}$, and thence the equation $\sqrt{rr + r\sqrt{4rr - xx}} = \frac{\overline{2rr - xx} \times \sqrt{4rrxx - x^4}}{r^3}$. Therefore, squaring, it will be $2rr + r\sqrt{4rr - xx} = \frac{4r^4 - 4r^2x^2 + x^4}{r^6} \times \overline{4rrxx - x^4}$; and squaring again, and ordering, we shall have $x^{14} - 16r^2x^{12} + 104r^4x^{10} - 352r^6x^8 + 660r^8x^6 - 672r^{10}x^4 + 336r^{12}x^2 - 63r^{14} = 0$. But this equation is divisible by $x^2 - 3r^2 = 0$. When the division is performed, we shall have $x^{12} - 13r^2x^{10} + 65r^4x^8 - 157r^6x^6 + 189r^8x^4 - 105r^{10}x^2 + 21r^{12} = 0$, which is not divisible by any divisor of two dimensions; wherefore the Problem seems to be of twelve dimensions. Therefore I resolve this Problem in another manner, retaining the same unknown quantity $x = AB = BF =$ &c. Because, in the triangles HCD, CDL, the angle CDH is common, and the angle at the circumference DCL, which insists upon the arch CD, the half of DA, these triangles will be similar, and therefore we shall have $DL = \frac{xx}{r}$, and $LH = r - \frac{xx}{r}$. But the angle DLC = DCH = EDH; wherefore the angle HLM, which is equal to the angle at the vertex DLC, will be equal to the angle EDH; whence the two right lines LM, DE, will be parallel, and the triangles HLM, HDE, will be similar, and therefore it will be $LM = \frac{rrx - x^3}{rr}$. But CL = CD = x, (the triangle LDC being similar to the isosceles triangle HDC,) and CL = MA, because the angles HLC, HMA, are equal, and therefore the triangles HLC, HMA, are equal and similar. Therefore $CA = 2x + \frac{rrx - x^3}{rr}$. And, because CA = CB, the equation will be $\frac{3rrx - x^3}{rr} = \sqrt{2rr + r\sqrt{4rr - xx}}$. And, by squaring, $9r^4x^2 - 6r^2x^4 + x^6 = 2r^6 + r^5\sqrt{4rr - xx}$. And, by squaring again, and ordering the terms, the equation will be $x^{10} - 12rrx^8 + 54r^4x^6 - 112r^6x^4 + 105r^8x^2 - 35r^{10} = 0$.

And thus I am arrived at another equation, which, because it is of an inferior degree to the first, must be multiplied by such a power of the unknown quantity, as is necessary to bring it to the same degree, so that it may be compared with that. Therefore, multiplying it by xx, it will be $x^{12} - 12r^2x^{10} + 54r^4x^8 - 112r^6x^6 + 105r^8x^4 - 35r^{10}x^2 = x^{12} - 13r^2x^{10} + 65r^4x^8 - 175r^6x^6 + 189r^8x^4 - 105r^{10}x^2 + 21r^{12}$. Now, subtracting the first from the second, it will be $x^{10} - 11r^2x^8 + 45r^4x^6 - 84r^6x^4 + 70r^8x^2 - 21r^{10} = 0$. Which, because it is of the tenth degree, being compared with the second equation found above, and subtracted from the same, will be $x^8 - 9r^2x^6 + 28r^4x^4 - 35r^6x^2 + 14r^8 = 0$, which may be divided by $xx - 2rr$; and making this division, we shall have at last this equation of the sixth degree, $x^6 - 7r^2x^4 + 14r^4x^2 - 7r^6 = 0$.

I have

I have proceeded in this way, to ſhow the uſe of the method. For otherwiſe, I might have gone more directly to the ſame equation, by comparing together the two values of the ſquares of CA, found in the two different ſolutions of the Problem; that is, $\frac{16r^6x^2 - 20r^4x^4 + 8r^2x^6 - x^8}{r^6}$ of the firſt, and $\frac{9r^4x^2 - 6r^2x^4 + x^6}{r^4}$ of the ſecond. For, making an equation between theſe two values, and taking away the terms that deſtroy one another, it will be $x^8 - 7r^2x^6 + 14r^4x^4 - 5r^6x^2 = 0$. And, dividing by x^2, it will be $x^6 - 7r^2x^4 + 14r^4x^2 - 5r^6 = 0$, as before. We might alſo, after a more compendious manner, have divided the equation firſt found by $x^6 - 6r^2x^4 + 9r^4x^2 - 5r^6 = 0$, and the ſecond by $x^4 - 5r^2x^2 + 5r^4 = 0$; and in each caſe we ſhould find the equation $x^6 - 7r^2x^4 + 14r^4x^2 - 7r^6 = 0$.

Yet the propoſed Problem is not of the ſixth degree, though it may ſeem to be ſuch, notwithſtanding all this care we take to depreſs it. To make this appear, we will retain the ſame compoſition of the figure, and make HI $= x$. Then it will be AI $= \sqrt{rr - xx} =$ IB, CI $= r + x$, CB $= \sqrt{rr + 2rx + xx + rr - xx}$ $= \sqrt{2rr + 2rx}$. Then, by purſuing the ſame way of arguing as before, we ſhall have CK $= \frac{2x\sqrt{rr - xx}}{r}$, HK $= \sqrt{\frac{r^4 - 4r^2x^2 + 4x^4}{rr}} = \frac{rr - 2xx}{r}$, CE $=$ 2CK $= \frac{4x}{r}\sqrt{rr - xx}$, CN $= \frac{4rrx - 8x^3}{r^3} \times \sqrt{rr - xx}$, CB $=$ 2CN $=$ $\frac{8rrx - 16x^3}{r^3}\sqrt{rr - xx}$. But we have before found CB $= \sqrt{2rr + 2rx}$. Therefore the equation will be $\sqrt{2rr + 2rx} = \frac{8rrx - 16x^3}{r^3} \times \sqrt{rr - xx}$.

Now I ſhall ſeek another equation after a different manner, but ſhall retain the ſame unknown quantity HI $= x$. By the ſame reaſoning as above, it will be DL $= \frac{4rr - 4xx}{r}$, LH $= r - \frac{4rr - 4xx}{r} = \frac{4xx - 3rr}{r}$, LM $= 2\sqrt{\frac{rr - xx}{rr}}$ $\times \overline{4xx - 3rr}$, CA $= 4\sqrt{rr - xx} + 2\sqrt{\frac{rr - xx}{rr}} \times \overline{4xx - 3rr}$; that is, by reduction, CA $= \frac{8xx - 2rr}{rr}\sqrt{rr - xx} =$ CB. Whence the equation $\sqrt{2rr + 2rx}$ $= \frac{8xx - 2rr}{rr}\sqrt{rr - xx}$; and laſtly, by equalling the *homogeneum comparationis* of each equation, it will be $\frac{8rrx - 16x^3}{r^3}\sqrt{rr - xx} = \frac{8xx - 2rr}{rr}\sqrt{rr - xx}$, which, being reduced, will be $8x^3 + 4rxx - 4rrx - r^3 = 0$, an equation only of the third degree.

180. When

180. When the methods above-described have been put in practice, if the equations cannot be depressed, but still remain above the second degree, we may proceed two ways in the solution of Problems, which arise to three or more dimensions. The way of least general use belongs only to equations of the third or fourth degree, and consists in resolving them by unravelling the analytical values of the unknown quantity, which therefore will present themselves under the form of cubick roots; which method is called *Cardan's* Rule. The The second way is more general, and of much more extensive use, and consists in finding the geometrical values of the unknown quantity, by means of the intersections of certain curve-lines, which are purposely introduced into the equation; that so the proposed Problem may be constructed. Solid Problems are solved by Cardan's rules, or by construction.

181. But, to begin with the analytical solution. I suppose the equations to be without the second terms, because they may always be reduced to such, if they are not such already. And all equations of the third degree, wanting the second terms, are comprehended under these four canonical formulæ. How by the four cases of Cardan's rule.

I. $x^3 - px - q = 0$. II. $x^3 + px - q = 0$.
III. $x^3 - px + q = 0$. IV. $x^3 + px + q = 0$.

Make $x = y + z$, then $px = py + pz$, and $x^3 = y^3 + 3y^2z + 3yz^2 + z^3$. And, substituting these values in the first equation, it will be $y^3 + 3y^2z + 3yz^2 + z^3 - py - pz - q = 0$. Of this we may form two equations, which are $3y^2z + 3yz^2 = py + pz$, and $y^3 + z^3 = q$. Dividing the first by $y + z$, we shall have $3yz = p$, or $y = \frac{p}{3z}$. This, substituted in the second, will give $\frac{p^3}{27z^3} + z^3 = q$, or $z^6 - qz^3 = -\frac{1}{27}p^3$. Whence, by the rule for affected quadraticks, $z^6 - qz^3 + \frac{1}{4}qq = \frac{1}{4}qq - \frac{1}{27}p^3$, and $z^3 = \frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$. Lastly, it will be $z = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. In the extraction of the square-root, I have taken only the positive sign, because the negative would bring no variation, and gives at last for the value of x the same quantity as the positive, as may be seen from the calculation. And it is to be understood in like manner in the other canonical equations. Now, because $y^3 + z^3 = q$, it will be therefore $y^3 = q - \frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$, and thence $y = \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. But it was at first $x = y + z$; therefore $x = \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. Hence it is seen, that the alternative of the signs, which was omitted, makes no variation.

182. The second equation $x^3 + px - q = 0$, making the same substitutions, will be $y^3 + 3y^2z + 3z^2y + z^3 + py + pz - q = 0$. From hence let the two equations be formed, $3y^2z + 3yz^2 = -py - pz$, and $y^3 + z^3 = q$. By the second case of the same rule.

 From

From the firſt, we have $3yz = -p$, or $y = -\frac{p}{3z}$, which, ſubſtituted in the ſecond, gives $-\frac{p^3}{27z^3} + z^3 = q$, or $z^6 - qz^3 = \frac{1}{27}p^3$. And therefore $z^3 = \frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}$, and $z = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$. But $y^3 + z^3 = q$, therefore $y = \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$, and $x = \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} + \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$.

The third caſe.

183. The third equation $x^3 - px + q = 0$, making the ſubſtitutions, will be $y^3 + 3y^2z + 3yz^2 + z^3 - py - pz + q = 0$. Let the two equations be formed, $3y^2z + 3yz^2 = py + pz$, and $y^3 + z^3 = -q$. From the firſt, we have $3yz = p$, or $y = \frac{p}{3z}$, which, ſubſtituted in the ſecond, gives $\frac{p^3}{27z^3} + z^3 = -q$, or $z^6 + qz^3 = -\frac{1}{27}p^3$; and therefore $z^3 = -\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$, and thence $z = \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. But $y^3 + z^3 = -q$; whence $y = \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$, and laſtly, $x = \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$.

The fourth and laſt caſe.

184. The fourth equation $x^3 + px + q = 0$, making the ſubſtitutions, will be $y^3 + 3y^2z + 3yz^2 + z^3 + py + pz + q = 0$. Forming the two equations, $3y^2z + 3yz^2 = -py - pz$, and $y^3 + z^3 = -q$, from the firſt we ſhall have $3yz = -p$, or $y = -\frac{p}{3z}$. This, ſubſtituted in the ſecond, gives $-\frac{p^3}{27z^3} + z^3 = -q$, or $z^6 + qz^3 = \frac{1}{27}p^3$, and therefore $z^3 = -\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}$, and thence $z = \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$. But $y^3 + z^3 = -q$; whence $y = \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$, and laſtly, $x = \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} + \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$.

Other expreſſions of the ſame roots.

185. The ſame roots or formulæ may be had, by putting $x = z \pm \frac{p}{3z}$, that is, $+\frac{p}{3z}$, if in the equation it be $-px$, and $-\frac{p}{3z}$, if it be $+px$ in the equation. Whence $x^3 = z^3 \pm pz + \frac{pp}{3z} \pm \frac{p^3}{27z^3}$. Make therefore the ſubſtitutions in the firſt canonical equation, and it will be $z^3 + \frac{p^3}{27z^3} - q = 0$, or $z^6 - qz^3 = -\frac{1}{27}p^3$, and $z^3 = \frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$, and then $z = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. Therefore, becauſe it was made $x = z + \frac{p}{3z}$, it will be $x = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \frac{p}{3\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}}$.

To

To reduce this to the ſame expreſſion found in the firſt manner, it will be ſufficient to multiply the numerator and denominator of the ſecond term of the *homogeneum comparationis* by $\sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$, and it will be $\frac{p\sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}}{3\sqrt[3]{\frac{1}{27}p^3}}$, that is, $\sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$, and therefore x will be the ſame as before. And the like may be obſerved in the other caſes.

186. It is evident that the values of the unknown quantity x, found by the firſt ſubſtitution of $x = y + z$, require the extraction of two different cubick roots, whereas the ſecond, by the ſubſtitution of $x = z \pm \frac{p}{3z}$, require the extraction of one only; and that the value by the ſecond and fourth canonical equation will always appear under a real form, becauſe the quantities under the quadratick radical are wholly poſitive. But that of the firſt and third will be under a real form, if $\frac{1}{4}qq$ be greater than $\frac{1}{27}p^3$; and under an imaginary form, when $\frac{1}{4}qq$ is leſs than $\frac{1}{27}p^3$. And this is called the Irreducible Caſe; but, notwithſtanding this, it does not follow, but that all it's roots are real. For all the three values in the firſt and third equation are real, when $\frac{1}{4}qq$ is leſs than $\frac{1}{27}p^3$. But when $\frac{1}{4}qq$ is greater than $\frac{1}{27}p^3$, in the firſt and third equation, and, in general, in the ſecond and fourth, the roots or values alone thus found are real, and the other two are imaginary.

To diſtinguiſh when theſe roots are real, and when imaginary.

As to the ſecond and fourth equation, this has been already demonſtrated at § 152, when they have the third term poſitive. Then, as to the firſt and third, when the third term is negative, each of theſe will have three real roots, which are $a, -b, -c$, or $-a, +b, +c$; and, becauſe the ſecond term is wanting, as is here ſuppoſed, it will be $a = b + c$, and the equation therefore, which ariſes from ſuch roots, will be of this form,

$$\begin{array}{l} x^3 - bbx \pm bc \times \overline{b+c} = 0. \\ \quad\; - bcx \\ \quad\; - ccx \end{array}$$

When b, c, are real quantities, then $\overline{b-c}^2$ will be a poſitive quantity; and therefore, if we put $bb - 2bc + cc = \mathrm{D}$, it will be alſo $bb + bc + cc = \mathrm{D} + 3bc$, and $\frac{\overline{bb+bc+cc}|^3}{27} = \frac{1}{27}\mathrm{D}^3 + \frac{1}{3}\mathrm{D}^2bc + \mathrm{D}bbcc + b^3c^3$. But beſides, it will be $bb + 2bc + cc = \overline{b+c}|^2 = \mathrm{D} + 4bc$, and therefore $\frac{1}{4}bbcc \times \overline{b+c}|^2 = \frac{1}{4}\mathrm{D}bbcc + b^3c^3$. And $\frac{1}{27}\mathrm{D}^3 + \frac{1}{3}\mathrm{D}^2bc + \mathrm{D}bbcc + b^3c^3$ is greater than $\frac{1}{4}\mathrm{D}bbcc$, and therefore it will alſo be greater than $\frac{1}{4}bbcc \times \overline{b+c}^2$, and therefore $\frac{1}{27} \times \overline{bb+bc+cc}|^3$ will be greater than $\frac{1}{4}bbcc \times \overline{b+c}|^2$. That is, the cube of the third part of the co-efficient of the third term, taken poſitively, is greater than the ſquare of half the laſt term; that is, $\frac{1}{27}p^3$ is greater than $\frac{1}{4}qq$. Therefore,

fore, if all the roots be real, the third term will always be negative, and befides, $\frac{1}{27}p^3$ will be greater than $\frac{1}{4}qq$. When it happens to be otherwife, two of the roots will be imaginary.

After the foregoing manner, having found one value for each equation, we fhall have the other two roots by dividing the propofed equation by this value; for the quotient will be an equation of the fecond degree, which may always be eafily refolved.

A compendium by the three cubick roots of unity.

187. But, if it fhall be thought convenient, the trouble of this divifion may alfo be fpared by confidering, that as unity itfelf has three cubick roots, which are 1, $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$; fo it may be underftood of any other quantity; of $\frac{1}{2}q+\sqrt{\frac{1}{4}qq+\frac{1}{27}p^3}$ for example, which, being multiplied into unity, it's three cubick roots will be $1\times\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}qq+\frac{1}{27}p^3}}$, $\overline{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}}$ into $\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}qq+\frac{1}{27}p^3}}$, and $\overline{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}}$ into $\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}qq+\frac{1}{27}p^3}}$.

Whence the three cubick roots of the firft equation $x^3-px-q=0$, by ordering them in a due manner, will be as follows: $x=\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}qq-\frac{1}{27}p^3}}+\sqrt[3]{\frac{1}{2}q-\sqrt{\frac{1}{4}qq-\frac{1}{27}p^3}}$, $x=\frac{-1+\sqrt{-3}}{2}\times\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}qq-\frac{1}{27}p^3}}+\frac{-1-\sqrt{-3}}{2}\times\sqrt[3]{\frac{1}{2}q-\sqrt{\frac{1}{4}qq-\frac{1}{27}p^3}}$, and $x=\frac{-1-\sqrt{-3}}{2}\times\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}qq-\frac{1}{27}p^3}}+\frac{-1+\sqrt{-3}}{2}\times\sqrt[3]{\frac{1}{2}q-\sqrt{\frac{1}{4}qq-\frac{1}{27}p^3}}$.

And, in fact, if we find the product of thefe three roots into each other, making, for brevity-fake, $\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}qq-\frac{1}{27}p^3}}=m$ and $\sqrt[3]{\frac{1}{2}q-\sqrt{\frac{1}{4}qq-\frac{1}{27}p^3}}=n$, the product of the laft, $x+\frac{1+\sqrt{-3}}{2}m+\frac{1-\sqrt{-3}}{2}n$ into the fecond, $x+\frac{1-\sqrt{-3}}{2}m+\frac{1+\sqrt{-3}}{2}n$ will be $xx+mx+nx+mm-mn+nn$, which, multiplied into the firft, $x-m-n$, will give $x^3-3mnx-m^3-n^3$; and, reftoring the values of m and n, it will be finally $x^3-px-q=0$, which is the equation propofed. Nor will it be otherwife in the other equations.

Example of this reduction.

188. The foregoing general formulæ being thus found, to apply them to the particular ufe of any given equations, it will be fufficient to compare the propofed equation to that of the four canonical equations which correfponds to it, thence to obtain the values of q and p; which, being fubftituted in the formula, will give the roots required.

Let

Let the equation be $x^3 + 2aax - 9a^3 = 0$. The correſponding one of the four canonical equations will be the ſecond, $x^3 + px - q = 0$; ſo that it will be $p = 2aa$, $q = 9a^3$. Then, making the ſubſtitution of theſe values inſtead of p and q, in the general expreſſion of the root of this ſecond equation, we ſhall have $x = \sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}} + \sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}}$, or, laſtly, $x = \sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{2219}{108}a^3}} + \sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{2219}{108}a^6}}$. The other two roots will be $x = \frac{-1+\sqrt{-3}}{2}\sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{2219}{108}a^6}} + \frac{-1-\sqrt{-3}}{2}\sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{2219}{108}a^6}}$, and $x = \frac{-1-\sqrt{-3}}{2}\sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{2219}{108}a^6}} + \frac{-1+\sqrt{-3}}{2}\sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{2219}{108}a^6}}$; the product of which roots will reſtore the propoſed equation.

189. But, without having recourſe to the general formulæ, particular equations may be ſolved independently of them, by making uſe of the given rule. Thus, for the equation $x^3 + 2aax - 9a^3 = 0$, making $x = y + z$, it will be $2aax = 2aay + 2aaz$, and $x^3 = y^3 + 3y^2z + 3yz^2 + z^3$; and, ſubſtituting theſe values in the propoſed equation, it will be changed into this other, $y^3 + 3zy^2 + 3z^2y + z^3 + 2aay + 2aaz - 9a^3 = 0$. Of this equation may be made theſe two, $3zy^2 + 3zzy = -2aay - 2aaz$, and $y^3 + z^3 = 9a^3$. From the firſt, by dividing by $y + z$, we have $3zy = -2aa$, or $y = -\frac{2aa}{3z}$; which, ſubſtituted in the ſecond, gives $-\frac{8a^6}{27z^3} + z^3 = 9a^3$, or $z^6 - 9a^3z^3 = \frac{8}{27}a^6$. And therefore $z^3 = \frac{9}{2}a^3 + \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}$, and $z = \sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}}$. But it is $y^3 + z^3 = 9a^3$, therefore $y^3 = \frac{9}{2}a^3 - \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}$, and $y = \sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}}$. But it is $y + z = x$, therefore $x = \sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}} + \sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}}$, the ſame as above. Examples without the formula.

Let the equation be $z^3 + 3az^2 - 5aaz + 2a^3 = 0$. Let the ſecond term be taken away, by making $z = x - a$, and there ariſes $x^3 - 8a^2x + 9a^3 = 0$. By comparing this with the third canonical equation, we ſhall have $p = 8aa$, $q = 9a^3$; whence, ſubſtituting theſe values in the general formula for the root, it will be $x = \sqrt[3]{-\frac{9}{2}a^3 + \sqrt{\frac{81}{4}a^6 - \frac{512}{27}a^6}} + \sqrt[3]{-\frac{9}{2}a^3 - \sqrt{\frac{81}{4}a^6 - \frac{512}{27}a^6}}$, that is, $x = \sqrt[3]{-\frac{9}{2}a^3 + \sqrt{\frac{139}{108}a^6}} + \sqrt[3]{-\frac{9}{2}a^3 - \sqrt{\frac{139}{108}a^6}}$. The like for the other two roots. And, becauſe it was made $z = x - a$, by ſubtracting the quantity a from each of the three roots, we ſhall have the roots of the propoſed equation.

Let the equation be $x^3 - 9a^2x + 2a^3 = 0$. This will correſpond to the third of the four canonical equations, and therefore it will be $p = 9a^2$, $q = 2a^3$; therefore, making a ſubſtitution of theſe values, inſtead of p and q in the general

expreſſion

expression of the root of that third equation, it will be $x = \sqrt[3]{-a^3 + \sqrt{-\frac{702}{27}a^6}} + \sqrt[3]{-a^3 - \sqrt{-\frac{702}{27}a^6}}$; which expression is imaginary, notwithstanding all the three roots are real; as the irreducible case requires.

Reduction of equations of the fourth degree.

190. In equations of the fourth degree, we may proceed after this manner. Let the canonical equation be $x^4 * + px^2 + qx - r = 0$, in which the second term is wanting; and if it had not been absent, it might have been taken away. Let this be transformed into a cubick equation, after the manner explained at § 167, by means of the two subsidiary formulæ, $x^2 + yx + z = 0$, and $x^2 - yx + u = 0$; and it will be transformed into

$$\begin{array}{l} y^6 + 2py^4 + ppy^2 - qq = 0. \\ \qquad\qquad\quad + 4ry^2 \end{array}$$

And the two subsidiary equations, by putting, instead of u and z, their values found from the comparison of the terms, will become $x^2 + yx + \frac{1}{2}p + \frac{1}{2}yy - \frac{q}{2y} = 0$, and $x^2 - yx + \frac{1}{2}p + \frac{1}{2}yy + \frac{q}{2y} = 0$. Now, as it is supposed that this equation has no divisor of two dimensions, the second term must be taken from it by the substitution of $yy = t - \frac{2}{3}p$, and then we shall have this new equation,

$$\begin{array}{lll} t^3 & -\frac{1}{3}ppt & -\frac{2}{27}p^3 = 0. \\ & + 4rt & -\frac{8}{3}pr \\ & & - qq \end{array}$$

Let this be compared with the first or second of the four canonical equations of § 181, according as $4r$ is lesser or greater than $\frac{1}{3}pp$, that we may have it's cube-root, which, for brevity-sake, we may call h. Whence it will be $t = h$; and, because it was made $yy = t - \frac{2}{3}p$, it will be $yy = h - \frac{2}{3}p$, and therefore $y = \sqrt{h - \frac{2}{3}p}$, which, for brevity, may be called g. In the two subsidiary formulæ put g instead of y, and gg instead of yy, and they will be $xx + gx + \frac{1}{2}gg + \frac{1}{2}p - \frac{q}{2g} = 0$, and $xx - gx + \frac{1}{2}gg + \frac{1}{2}p + \frac{q}{2g} = 0$; the roots of which are $x = -\frac{1}{2}g \pm \sqrt{\frac{q}{2g} - \frac{1}{2}p - \frac{1}{4}gg}$ of the first, and $x = \frac{1}{2}g \pm \sqrt{-\frac{q}{2g} - \frac{1}{2}p - \frac{1}{4}gg}$ of the second. And, restoring the value of $g = \sqrt{h - \frac{2}{3}p}$, they will be $x = -\frac{1}{2}\sqrt{h - \frac{2}{3}p} \pm \sqrt{\frac{q}{2\sqrt{h - \frac{2}{3}p}} - \frac{1}{3}p - \frac{1}{4}h}$, and $x = \frac{1}{2}\sqrt{h - \frac{2}{3}p} \pm \sqrt{\frac{-q}{2\sqrt{h - \frac{2}{3}p}} - \frac{1}{3}p - \frac{1}{4}h}$, the four roots of the proposed equation $x^4 * + px^2 + qx - r = 0$.

Let the equation be $x^4 * - 86aax^2 + 600a^3x - 851a^4 = 0$. This being compared with the foregoing canonical equation, we shall have $p = -86aa$, $q = 600a^3$, $r = 851a^4$. Therefore the transformed cubical equation will be $y^6 - 172aay^4 + 10800a^4y^2 - 360000a^6 = 0$. Now, because this is divisible by

by $y^2 - 100a^2 = 0$, without resolving it by the rules of cubick equations, as we know already the root to be $yy = 100aa$, and $y = 10a$; substitute these values instead of y and yy, as also the values of p, q, in the two subsidiary equations, they will be $x^2 + 10ax - 23aa = 0$, and $x^2 - 10ax + 37aa = 0$, and their roots are $x = -5a \pm \sqrt{48aa}$, and $x = 5a \pm \sqrt{-12aa}$, which are therefore the four roots of the proposed equation. This example is inserted only to show the use of the method; for the given equation may be reduced to two of two dimensions, after the way already explained in it's place.

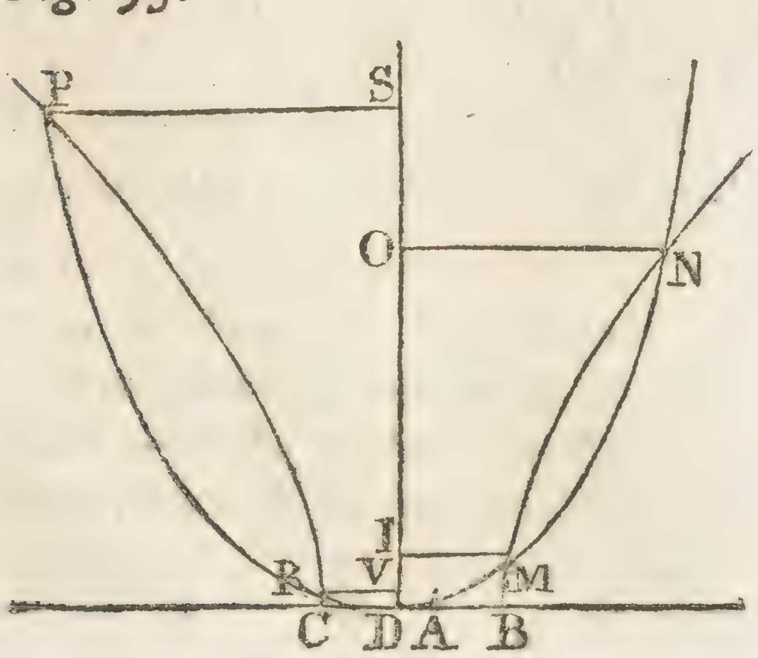
191. This method of resolving equations can be of use only in arithmetical questions, and not in geometrical: because, in this way, we have the value of the unknown quantity expressed by a cube-root, which it is supposed cannot be actually extracted; for, otherwise, the equation would have a divisor, and would not be of the degree it seems to be. Now, to find this cube-root geometrically cannot be done otherwise than by the intersection of curve-lines; which is the second manner, and the general one which I have mentioned before, at § 180.

How equations may be resolved geometrically by a combination of *loci*.

This method consists in introducing a new unknown quantity into the equation, by which we shall have two equations, each of which contains both the unknown quantities, and both of them together all the known quantities of the proposed equation. These two equations are two *loci geometrici*, which are therefore to be constructed; the intersections of which determine the geometrical values, or the roots of the equation proposed. And the reason of this is manifest. For, as from the combination of two places, or from two indeterminate equations, by putting in one of these, instead of one of the two unknown quantities, it's value given by the other equation, there arises a determinate equation, which determinate equation may be resolved into two indeterminates.

Let there be given the two equations $ax = zz$, and $xx - 5zz + 2az + 3aa = 0$. If from the first, for example, we derive the value of $x = \frac{zz}{a}$, and substitute it in the second, there will arise the determinate equation $z^4 - 5aazz + 2a^3z + 3a^4 = 0$, of the fourth degree. Then, taking the *locus* to the parabola $ax = zz$, if we make the substitution of the value of zz in the equation $z^4 - 5aazz + 2a^3z + 3a^4 = 0$, there will arise the second *locus* $aaxx - 5aaz^2 + 2a^3z + 3a^4 = 0$, or $x^2 - 5z^2 + 2az + 3aa = 0$. To construct this second *locus*, with centre A (Fig. 95.) and transverse axis CB $= \frac{8}{5}a$, and with the parameter $= 8a$, let there be described the two opposite hyperbolas BN, CP, which shall be the *locus* of the equation $x^2 - 5z^2 + 2az + 3a^2 = 0$, taking the abscis z from the point D, which is distant from the centre A by the quantity $\frac{1}{5}a$ towards the vertex C.

Fig. 95.



Rightly to combine this with the firſt *locus* $ax = zz$, it is neceſſary that the origin and the axis, of the unknown quantity x, may be in common to both the *loci*. And therefore at the vertex D, with the parameter $= a$, upon the axis DO, parallel to the conjugate axis of the oppoſite hyperbolas, the parabola of the firſt equation $ax = zz$ ſhould be deſcribed. This will meet the two hyperbolas in the four points M, N, R, P, from which drawing the perpendiculars MI, NO, RV, PS, to the axis DO, they will be the four values of z, that is, the four roots of the equation $z^4 * - 5aaz^2 + 2a^3z + 3a^4 = 0$. The two IM, ON, will be poſitive, and the other two VR, SP, will be negative. For, as z of the determinate equation, (that is to ſay, every one of the roots of the ſame,) ought to be common to both the *loci*, this can happen only in the points M, N, R, P, in which theſe two *loci* interſect each other. Therefore the right lines MI, NO, RV, SP, which expreſs z, will be the four roots of the determinate equation propoſed.

When two of the roots will be equal, when nothing, when imaginary.

192. Hence it is plain, that the nearer the points M, N, approach to each other, ſo much the leſs will be the difference of the ordinates IM, ON. So that when one point falls on another, (in which caſe the two curves will no longer cut but touch each other,) the two ordinates become equal, or the equation will have two equal roots. Alſo, if the curves cut each other at the vertex, in which place the ordinate is nothing, the equation will have one of it's roots equal to nothing. And laſtly, if the two curves neither cut nor touch in any point, the roots of the propoſed equation will be imaginary or impoſſible.

The *loci* ſhould be ſuch, as will ſupply the ſimpleſt conſtruction.

193. Now, in the introduction of the new unknown quantity, it ſhould be endeavoured, that it may be done in ſuch a manner, as that the two *loci* may be the ſimpleſt poſſible, in reſpect of the degree of the propoſed equation. That is to ſay, if the equation be of the third or fourth degree, the two *loci* ſhould be of the ſecond, that is, conic ſections. And it might be convenient, as any one would think, that one of them ſhould always be a circle, as being the ſimpleſt curve. But it ought to be conſidered, that, by determining one of the *loci* to be a circle, the equation to the other *locus* in many caſes may become perplexed ; and therefore in ſuch caſes I ſhould prefer any other *locus* before the circle, if it would afford a greater ſimplicity. If the equation be of the fifth or ſixth degree, the two *loci* may be one of the ſecond, and the other of the third. If it be of the ſeventh or eighth, they ſhould be one of the ſecond, and one of the fourth ; or two of the third, firſt reducing that of the eighth to the ninth. And ſo on, obſerving the ſame analogy.

Taking, therefore, this equation of the fourth degree, $x^4 + 2bx^3 + acx^2 - a^2dx - a^3f = 0$, aſſume the equation (I.) $xx + bx = ay$, and, by ſquaring, it will be $x^4 + 2bx^3 + b^2x^2 = a^2y^2$, and therefore $x^4 + 2bx^3 = a^2y^2 - b^2x^2$. In the propoſed equation let this value be ſubſtituted inſtead of $x^4 + 2bx^3$, and there will ariſe this other equation, (II.) $yy - \frac{b^2x^2}{a^2} + \frac{cx^2}{a} - dx - af = 0$.

Now,

Now, putting the value of xx obtained from the firſt equation, that is, $ay - bx$, in the ſecond term of this, and letting the third term alone, there will ariſe (III.) $yy - \frac{bb}{a}y + \frac{b^3}{a^2}x + \frac{c}{a}x^2 - dx - af = 0$. Or, ſubſtituting the value of xx in the third term of the ſame equation, letting the ſecond term alone, there will ariſe (IV.) $yy - \frac{bb}{aa}xx + cy - \frac{bc}{a}x - dx - af = 0$. And in this, putting the value of xx, it will be (V.) $yy + cy - \frac{bb}{a}y - \frac{bc}{a}x - dx - af = 0$. Laſtly, if from this be ſubtracted the firſt made equal to nothing, or $xx + bx - ay = 0$, and then adding it to the ſame, there will ariſe from the firſt operation (VI.) $yy + cy - \frac{bb}{a}y + ay - xx - bx - \frac{bc}{a}x + \frac{b^3}{a^2}x - dx - af = 0$; and from the ſecond, (VII.) $yy + cy - \frac{bb}{a}y - ay + xx + bx - \frac{bc}{a}x + \frac{b^3}{a^2}x - dx - af = 0$.

194. It is plain, that the firſt equation is a *locus* to the *Apollonian* parabola. To diſtinguiſh the reſt, we muſt make uſe of the reductions explained at § 127, 128, by which we ſhall find, that the ſecond will be a *locus* to the parabola, when it is $ac = bb$; to the ellipſis, when ac is greater than bb; and, finally, to the hyperbola, when ac is leſs than bb. The third will be to an ellipſis, which will degenerate into a circle, when it is $c = a$, and the co-ordinates are at right angles. The fourth will be to an hyperbola, which beſides will be equilateral, if it is $b = a$. The fifth will be to a parabola. The ſixth will be to the equilateral hyperbola. The ſeventh will be to the circle, when the angle of the co-ordinates is a right angle. To diſtinguiſh theſe *loci*.

From hence we may make choice of ſuch a combination of the two *loci*, for the conſtruction of the propoſed Problem, as ſhall be thought moſt convenient.

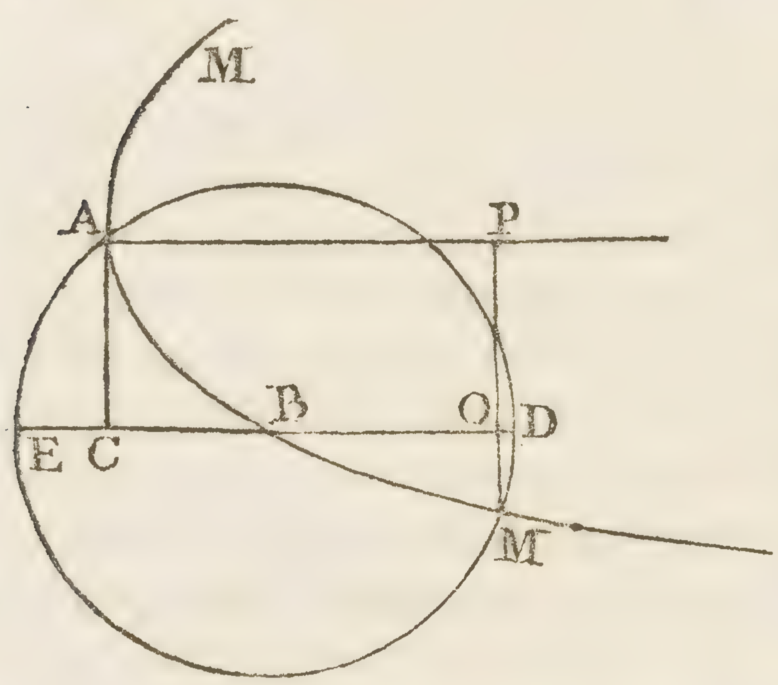
195. If the ſecond term of the propoſed equation had been negative, we ſhould have made $xx - bx = ay$; and the equations thence ariſing would have been the ſame as before, only changing the ſign of thoſe terms, in which the letter b is of odd dimenſions. And if the propoſed equation had at firſt been without the ſecond term, I ſhould have taken $xx = ay$. Therefore, expunging the terms in which b is found in the other equations, they would have been ſuch as this caſe requires. Cautions to be obſerved.

196. In the propoſed equations, the ſecond term being $\pm 2bx^3$, we ſhould take the *locus* to the parabola $xx \pm bx = ay$, rather than $xx = ay$; becauſe thus the other *loci* which ariſe have not the rectangle xy, and therefore are conſtructed with the more eaſe. Conſtruction of a cubic equation, for example, by a parabola and a circle.

EXAMPLE I.

Let the equation be of the third degree, $x^3 - aax + 2a^3 = 0$. Let it be multiplied by $x = 0$, to reduce it to the fourth degree; whence it will be $x^4 - aax^2 + 2a^3x = 0$; which is required to be conſtructed by means of a parabola and a circle. As the ſecond term is wanting, make $xx = ay$, a *locus* to the parabola. Then ſubſtituting, inſtead of x^4 and x^2, their values $aayy$ and ay, it will be $yy - ay + 2ax = 0$; to which adding the firſt equation $xx - ay = 0$, we ſhall have the equation $yy - 2ay + 2ax + xx = 0$, which is a *locus* to the circle.

Fig. 96.



With radius $BD = \sqrt{2aa}$ let the circle ADME be deſcribed, and make $BC = a$, and alſo the ordinate $CA = CB = a$. From the point A drawing the indefinite line AP parallel to ED, and on it taking the abſciſſes $AP = y$, and making the ordinate $PM = x$, this will be the *locus* of the equation $yy - 2ay + 2ax + xx = 0$. Upon the axis AP, on which are taken the y's, with vertex A let the *Apollonian* parabola MAM of the equation $xx = ay$ be deſcribed, which ſhall cut the circle in two points A, M; from whence the ordinates being drawn, they ſhall be the the real roots of the equation $x^4 * - aax^2 + 2a^3x = 0$, and two will be imaginary.

But at the point A the ordinate is nothing, and therefore one of the roots will be $x = 0$, as it ought to be; it being now introduced by multiplying the propoſed equation by $x = 0$. Therefore PM will be the real negative root of the equation $x^3 - aax + 2a^3 = 0$, and the other two will be imaginary. If I had multiplied the propoſed equation by x equal to ſome quantity, the circle would have cut the parabola in two points out of the vertex, one of which would have given me the introduced root, and the other that of the propoſed equation.

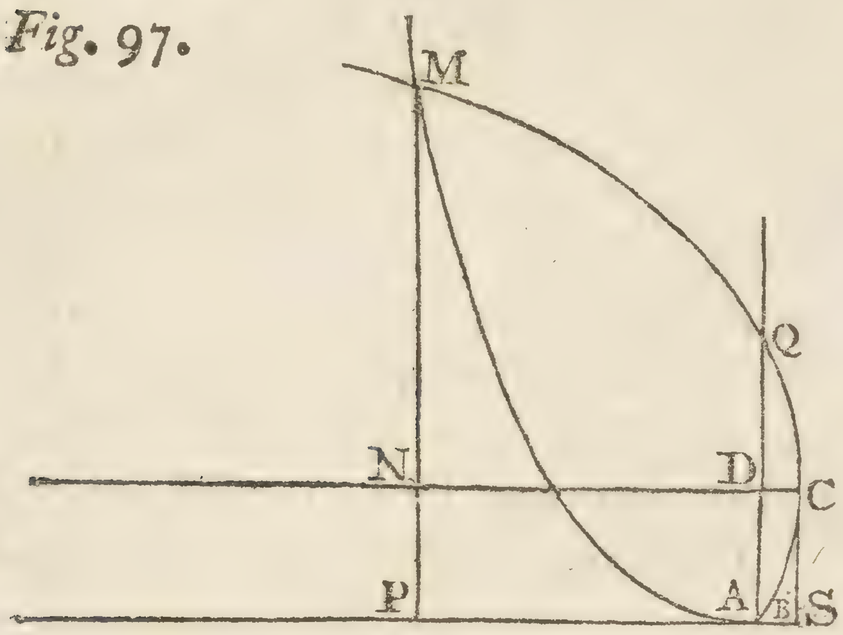
Now, to ſhow that PM is one of the roots of the equation $x^4 - aax^2 + 2a^3x = 0$, it may be conſidered, that, from the nature of the circle, it is $EO \times OD = OMq$. But $OM = -x - a$, $EO = y + \sqrt{2aa} - a$, and $OD = a - y + \sqrt{2aa}$. Therefore $xx + 2ax + aa = aa + 2ay - yy$. But, by the equation of the parabola AM, it is $xx = ay$, and therefore $\frac{x^4}{aa} = yy$.

Then

Then ſubſtituting theſe values of y and yy, and reducing the equation to nothing, it will be $x^4 - aax^2 + 2a^3x = 0$, which is the very equation of the fourth degree, whoſe roots we were to extract.

197. If we would conſtruct the equation $x^4 - aax^2 + 2a^3x = 0$ by means of two parabolas, it would be convenient to make uſe of the equation found above, $yy - ay + 2ax = 0$; and the *locus* of this, together with the parabola of the equation $xx = ay$, might determine the roots required. —By two parabolas.

Fig. 97.



Therefore, with parameter $= 2a$, let there be deſcribed the parabola MCA, in which make CD $= \frac{1}{8}a$. And letting fall DA $= \frac{1}{2}a$, which will meet the parabola in the point A, and through that point drawing the indefinite line AP parallel to the axis CD; and taking the abſciſs x from the point A, poſitive towards B and negative towards P, and the ordinates PM $= y$, this will be the *locus* of the equation $yy - ay + 2ax = 0$. Then with vertex A, to the axis AQ, let the other parabola MAS of the equation $xx = ay$ be deſcribed; this will cut the firſt in the points A, M. And letting fall the perpendicular MP, it will give the negative root AP of the propoſed equation. And becauſe at the point A the perpendicular is nothing, therefore there is no other root; juſt as it ought to be, the propoſed equation being multiplied by $x = 0$.

For, in the parabola MCA, it being CN $= -x + \frac{1}{8}a$, and NM $= y - \frac{1}{2}a$, it will be, by the property of this parabola, $\frac{1}{4}aa - 2ax = yy - ay + \frac{1}{4}aa$; and ſubſtituting the values of y and yy, which are given by the firſt equation to the parabola MAS, that is, $xx = ay$, and ordering the equation, we ſhall have at laſt $x^4 * - aax^2 + 2a^3x = 0$, which is the equation of the fourth degree, of which the roots were required.

198. Now, if I had intended to have made uſe of the parabola, and of the equilateral hyperbola, it would have ſufficed, from the ſame equation $yy - ay + 2ax = 0$, to have ſubtracted the firſt equation $xx - ay = 0$, and the equation $yy + 2ax - xx = 0$ would have ariſen from thence, which is a *locus* to the equilateral hyperbola; which, being conſtructed, would have given me the roots required, by means of it's interſections with the parabola of the equation $xx = ay$. —By a parabola and an equilateral hyperbola.

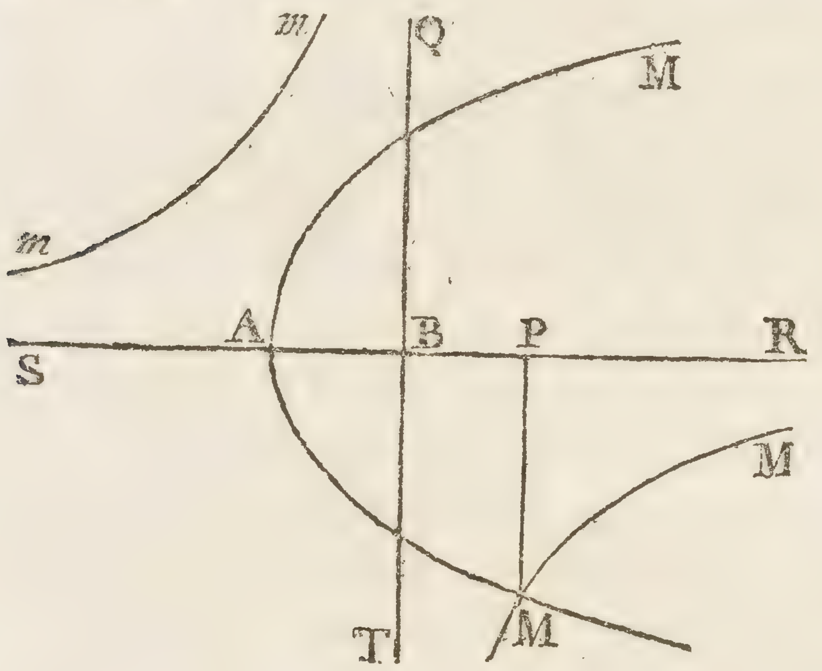
199. Finally, if I had deſired to ſolve the Problem by the circle and the hyperbola, I ſhould have conſtructed the third equation $yy - 2ay + 2ax + xx = 0$, a *locus* to the circle, and the fourth equation $yy + 2ax - xx = 0$, a *locus* to the hyperbola, as is ſeen before; the interſections of which *loci* would have given me the roots required. —By a circle and hyperbola.

 200. But,

These equations constructed by various *loci*, with examples.

200. But, without multiplying by x the equation proposed, $x^3 - aax + 2a^3 = 0$, we might have constructed it after the following manner, when we do not choose to introduce one *locus* rather than another. Make therefore $xx = ay$, and, instead of xx, put it's value ay in the equation, and there will arise the equation $xy - ax + 2aa = 0$, a *locus* to the hyperbola between it's asymptotes.

Fig. 98.



Therefore let the two indefinite right lines SR, QT, cut each other at right angles, and let these be the asymptotes of the two hyperbolas MM, *mm*, having the constant rectangle $-2aa$; taking the absciffes from the point A, distant from the point B by the quantity a. At the vertex A, to the axis AR, with the parameter $= a$, let the parabola of the first equation $xx = ay$ be described; it will cut the hyperbola MM in the point M. Then drawing the ordinate PM, it will be the real and negative root of the proposed equation.

For, by the property of the hyperbola MM, it will be $BP \times PM = -2aa$, that is, $xy - ax = -2aa$. And, by the property of the parabola AM, we shall have $y = \frac{xx}{a}$. Therefore, instead of y, substituting it's value, and ordering the equation, it will be $x^3 - aax + 2a^3 = 0$, the equation proposed.

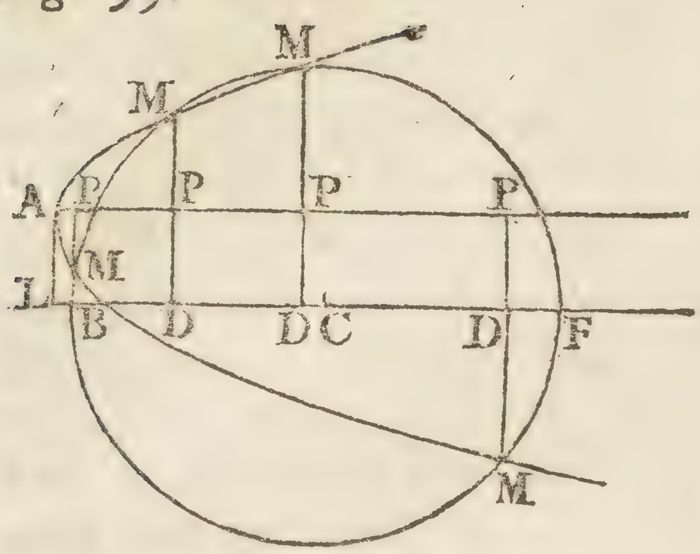
In general, all equations of the third degree may always be constructed after this manner, without being reduced to the fourth: by a parabola, and an hyperbola between the asymptotes.

EXAMPLE II.

Let there be given the equation of the fourth degree, $z^4 * - 5a^2z^2 + 2a^3z + 3a^4 = 0$, which is to be constructed by means of a parabola and a circle. Take the equation $ax = zz$, square it, and in the equation proposed, instead of z^4 and z^2, substitute their values, and there will arise a second equation, $xx - 5ax + 2az + 3aa = 0$, from whence subtracting and then adding the first equation $zz - ax = 0$, we shall have, in the first case, a third equation, $xx - 4ax + 2az + 3aa + zz = 0$; and in the second case, a fourth equation, $xx - 6ax + 2az + 3aa + zz = 0$; which is a *locus* to the circle, and therefore I shall make use of it to construct the proposed equation of the fourth degree.

With

Fig. 99.



With radius $= \sqrt{7aa}$ let there be described a circle BMF, and from the centre C towards B taking the line CL $= 3a$, and from the point L make LA $= a$, perpendicular to the diameter, from the point A draw the indefinite right line AP parallel to the diameter BF; it will be AP $= x$, and the corresponding ordinates in the circle PM $= z$. And therefore A will be the vertex, and AP the axis of the parabola of the equation $ax = zz$. Whence, with the vertex A, axis AP, and parameter $= a$, describing the parabola AM, it will meet the circle in four points M, from whence drawing the perpendiculars PM to the axis AP, they will be the roots of the proposed equation, two being positive and two negative.

For, producing PM to D, if there be occasion, it will be, by the nature of the circle, BD $\times$ DF $=$ DMq. But DM $= z + a$, BD $= x - 3a + \sqrt{7aa}$, and DF $= -x + 3a + \sqrt{7aa}$. Therefore $zz + 2az + aa = -xx + 6ax - 2aa$; but, by the nature of the parabola AM, it is $ax = zz$, and $xx = \frac{z^4}{aa}$. Therefore, making a substitution of these values, and ordering the equation, and bringing the terms all to one side, it will be $z^4 - 5aaz^2 + 2a^3z + 3a^4 = 0$, which is the equation proposed.

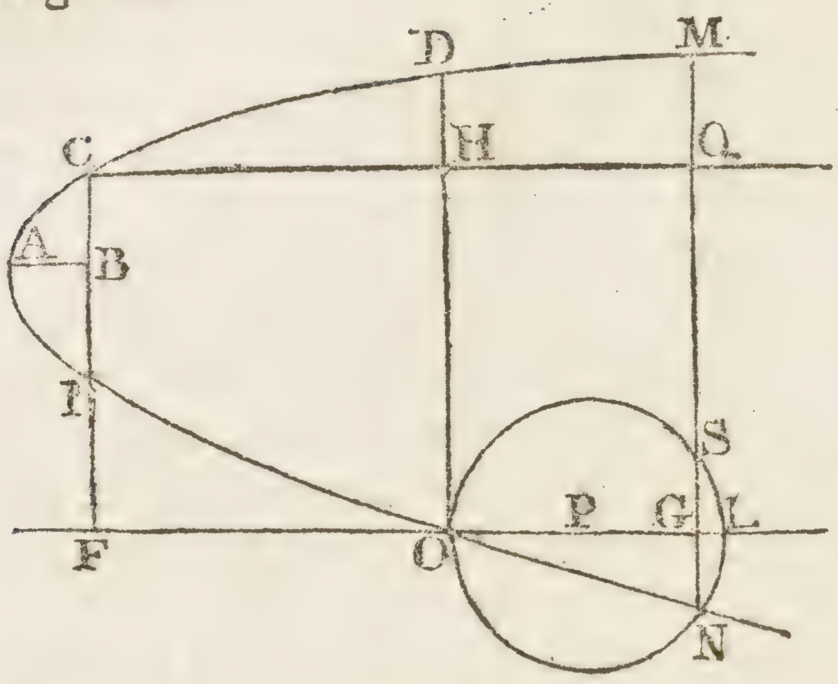
EXAMPLE III.

Let there be given an equation of the third degree, $x^3 - 3aax + 5a^3 = 0$, and let it be multiplied by $x + 2a$, that it may be reduced to one of the fourth degree; it will be $x^4 + 2ax^3 - 3aax^2 - a^3x + 10a^4 = 0$.

Take the equation to a parabola $xx + ax = ay$, which, by squaring, will become $x^4 + 2ax^3 + aax^2 = aayy$. Let the value of it's two first terms, $x^4 + 2ax^3$, that is, $aayy - aaxx$, be substituted in the equation, and there will arise (II.) $yy - 4xx - ax + 10aa = 0$. And in this, instead of xx, substituting it's value $ay - ax$, there arises (III.) $yy - 4ay + 3ax + 10aa = 0$; from thence subtracting the first, $xx + ax - ay = 0$, and also adding it, there will arise these two equations, (IV.) $yy - 3ay + 2ax + 10aa - xx = 0$ in the first case, and (V.) $yy - 5ay + 4ax + 10aa + xx = 0$ in the second case. I shall make use of the first *locus*, and also of the last.

For

Fig. 100.



For the construction of the last, let the circle OSN be described, with radius OP $= \frac{1}{2}a$; and, producing it to F, that it may be OF $= 2a$, and at the point F erecting the perpendicular FC $=$ FO $= 2a$, draw the indefinite right line CQ parallel to FP. Taking any line whatever, CQ $= y$, the corresponding negative ordinates, QS, QN, will represent x, and the circle will be the *locus* of the fifth equation. Now take in FC the line CB $= \frac{1}{2}a$, and from the point B draw the perpendicular BA $= \frac{1}{4}a$. Then with vertex A, and with parameter $= a$, let the parabola NAM be described, which shall be the *locus* of the first equation, taking the abscisses y on the right line CQ. From the points O, N, in which the parabola cuts the circle, raising the perpendiculars OH, NQ, these will be the two real negative roots of the equation, $x^4 + 2ax^3 - 3a^2x^2 - a^3x + 10a^4 = 0$, of the fourth degree which was proposed.

And because OH, taken negative, is equal to $2a$, which is the root introduced by the multiplication of the given equation into $x + 2a$, NL will be the real negative root of the proposed equation $x^3 - 3aax + 5a^3 = 0$, the other two roots being imaginary.

For, by the property of the circle OSL, it will be OG $\times$ GL $=$ GNq. But OG $= y - 2a$, GL $= 3a - y$, and GN $= -2a - x$. Therefore, making the substitutions, it will be $xx + 4ax + 10aa + yy - 5ay = 0$. But, from the equation to the parabola NAM, it will be $y = \frac{xx + ax}{a}$, and $yy = \frac{x^4 + 2ax^3 + aaxx}{aa}$; then substituting these values of y and yy in the equation to the circle, it will be at last $x^4 + 2ax^3 - 3aaxx - a^3x + 10a^4 = 0$, as it ought to be.

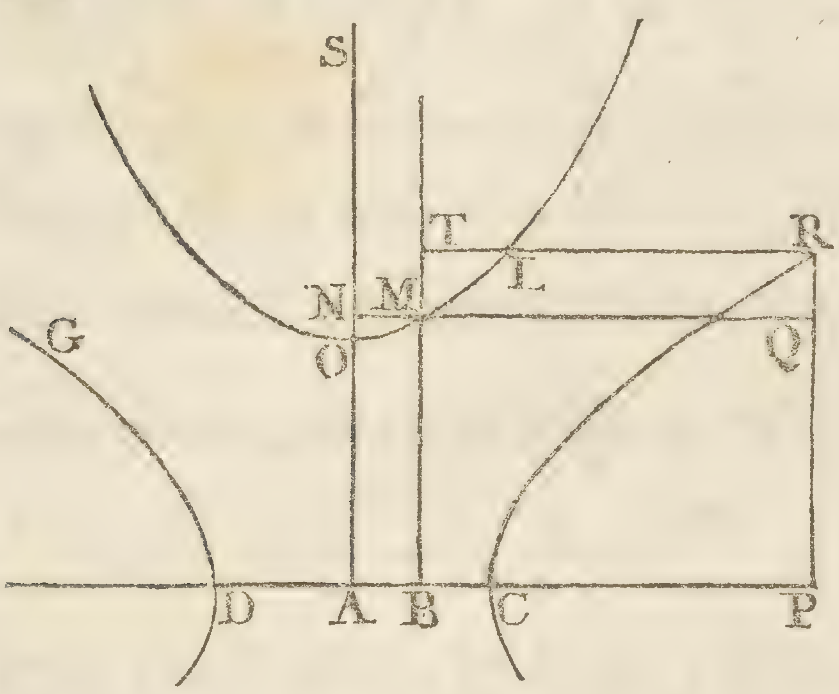
EXAMPLE IV.

Let the equation be $x^6 - 4aax^4 - 8a^3x^3 + 8a^4x^2 + 32a^6 = 0$; and because it is divisible by $x^2 - 4ax + 4aa$, and the quotient $x^4 + 4ax^3 + 8a^2x^2 + 8a^3x + 8a^4 = 0$ is an equation of the fourth degree, which we thus construct; take the equation $xx + 2ax = ay$, of which finding the square $x^4 + 4ax^3 + 4a^2x^2 = a^2y^2$, and, instead of $x^4 + 4ax^3$, substitute it's value $aayy - 4aaxx$ in

in the equation, and there will arise (II.) $yy + 4xx + 8ax + 8aa = 0$; in which, if we put the value of xx, or $ay - 2ax$, there will arise (III.) $yy + 4ay + 8aa = 0$, from which, if we subtract the first, there arises (IV.) $yy + 5ay + 8aa - xx - 2ax = 0$; and lastly, if we add the first to the third, it will be (V.) $yy + 3ay + 8aa + xx + 2ax = 0$.

The second *locus* is imaginary. The third is a determinate equation, but it's roots are imaginary. The fifth *locus* is also imaginary. But the fourth *locus* is real, and is to an equilateral hyperbola.

Fig. 101.



To the axis DC $= \sqrt{11aa}$, with centre A, let there be described the hyperbolas CR, DG. Take AB $= a$, and let the indefinite perpendicular BM be raised, in which take BM $= \frac{5}{2}a$; and from the point M let there be drawn MQ parallel to the axis DC. Taking the x's from the point M upon MQ, the corresponding QR or MT will be the y's, and the curve is the *locus* of the fourth equation. Producing QM to N, and making MN $= a$, and drawing NA to the centre of the hyperbola, take NO $= a$, and with vertex O, parameter $= a$, to the axis OS let the parabola OM be described, which will pass through the point M. Then taking the y's on MT, and the corresponding ordinates TL $= x$, this will be the *locus* of the first equation $xx + 2ax = ay$. But now, as these two *loci* can never interfect each other, as is evident, all the four roots of the equation $x^4 + 4ax^3 + 8a^2x^2 + 8a^3x + 8a^4 = 0$ will be imaginary. Whence the proposed equation $x^6 - 4a^2x^4 - 8a^3x^3 + 8a^4x^2 + 32a^6 = 0$ is found to have only two real roots, which are equal to each other, being each equal to $2a$.

201. But if, besides, we should be willing to construct equations of the third and fourth degree, not only by the help of conical *loci*, which are to be thus found, but of such of them as may be given, or similar to given, *loci*; which may be of use when a conic section is given in the state of a Problem: It may be done after the following manner, supposing, however, that the equations of the third degree are reduced to the fourth, and that these are freed from their second term, if they have any. —By given *loci*, or such as are similar to given.

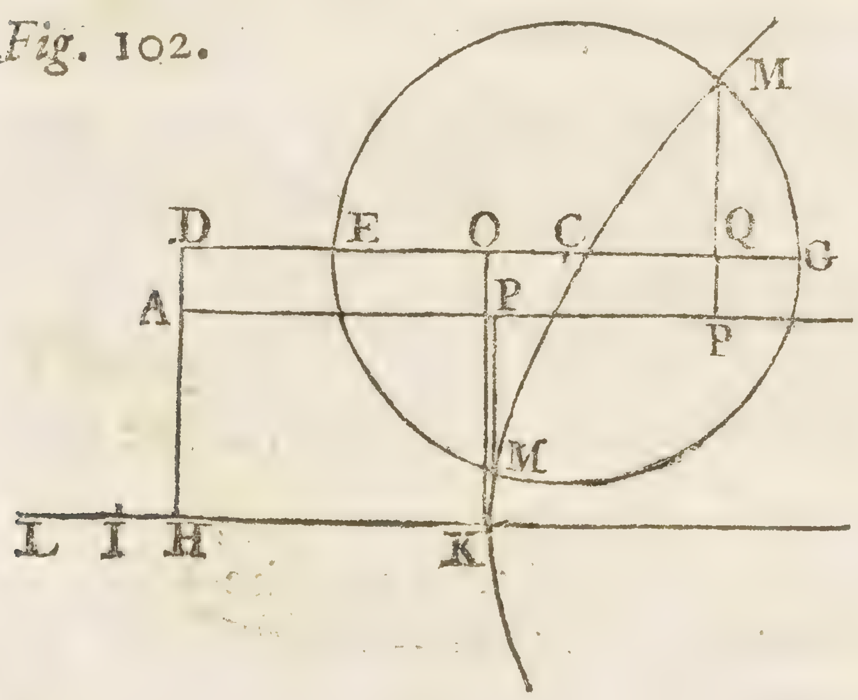
Yet I must here observe, that though, for the most part, it may be better to be determined to this conical *locus* which already enters into the Problem; yet we should always have it in view, that the use of this given *locus* ought not to supersede a greater simplicity of construction. For, in this case, without any regard to the given *locus*, it may be better to introduce two new *loci*.

Being willing then to make ufe of given *loci*, or fuch as are fimilar to thofe that are given, the artifice confifts in introducing two indeterminate or general quantities into the equation, and to determine them afterwards as occafion may require. Therefore let the equation be $z^4 + abz^2 - aacz + a^3d = 0$. Make $z = \frac{ax}{f}$, in order to introduce the firft indeterminate f. Making the fubftitutions, it will be $x^4 + \frac{bffx^2}{a} - \frac{f^3cx}{a} + \frac{f^4d}{a} = 0$. Let us take the firft *locus* (I.) $x^2 - fy = 0$; and, fubftituting the values of x^2 and x^4, there will arife the fecond *locus* (II.) $y^2 + \frac{bf}{a}y - \frac{fc}{a}x + \frac{ffd}{a} = 0$. To this let be added the firft, and we fhall have (III.) $x^2 - fy + yy + \frac{bf}{a}y - \frac{fc}{a}x + \frac{ffd}{a} = 0$. Now, to introduce a fecond indeterminate g, let the firft *locus* be multiplied by $\frac{g}{a}$, and we fhall have $\frac{gx^2 - gfy}{a} = 0$; which, added to the fecond, will give (IV.) $y^2 + \frac{bf}{a}y - \frac{fc}{a}x + \frac{ffd}{a} + \frac{g}{a}x^2 - \frac{gf}{a}y = 0$; and, being fubtracted, will give (V.) $y^2 + \frac{bf}{a}y - \frac{fc}{a}x + \frac{ffd}{a} + \frac{gf}{a}y - \frac{g}{a}x^2 = 0$.

The firft *locus* and the fecond are to a parabola; the third to the circle, when the co-ordinates are at right angles; the fourth to the ellipfis; and the fifth to the hyperbola.

Now, let it be required, for example, to conftruct the equation by means of a given circle and a given hyperbola. Let us therefore affume the third and fifth *loci*; and as to the third, with radius $CG = \frac{f}{2a}\sqrt{cc - 4ad + bb - 2ab + aa}$, let the circle EMG be defcribed, and, taking $CD = \frac{fc}{2a}$, from the point D let fall the perpendicular $DA = \frac{af - bf}{2a}$, (fuppofing a to be greater than b; for it muft be raifed the contrary way, when b is greater than a,) then from the point A, on the right line AP parallel to DG, taking the abfciffes $AP = x$, the correfponding PM will be the y, and the circle EMG will be the *locus* of the equation $x^2 - fy + y^2 + \frac{bf}{a}y - \frac{fc}{a}x + \frac{ffd}{a} = 0$.

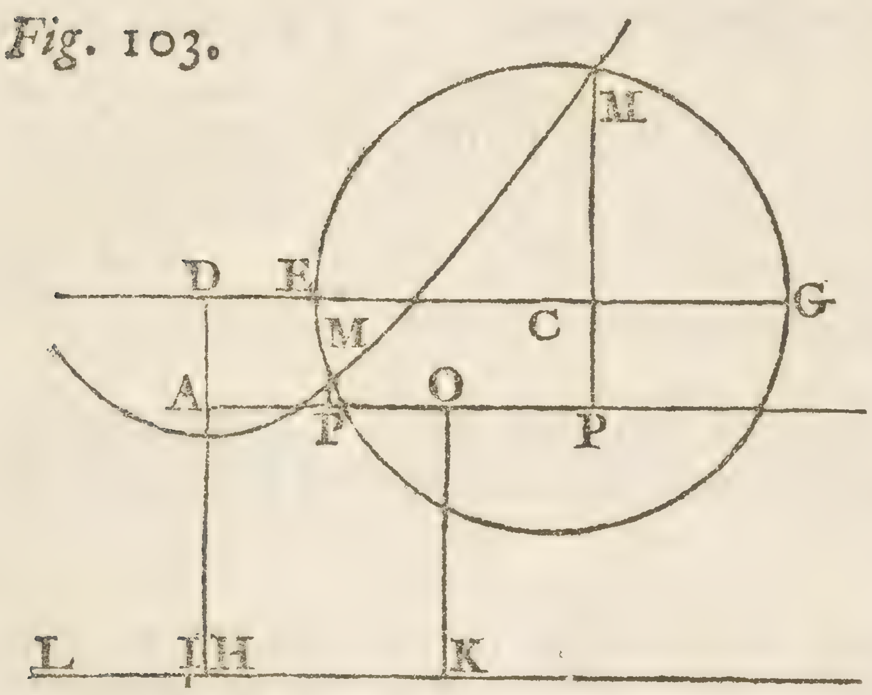
Fig. 102.



As

As to the fifth *locus*; to construct it and combine it with the circle, through the point A, the origin of x, produce the right line DA to H, so that it may be $AH = \frac{gf + bf}{2a}$; and through the points A, H, draw AP, HK, parallel to DG. On HK, towards the point L, set off the portion $HI = \frac{fc}{2g}$, and with centre I, transverse axis $LK = \frac{f}{ga}\sqrt{aacc + 4a^2gd - ab^2g - ag^3 - 2abg^2}$, (supposing $cc + 4dg$ to be greater than $\frac{bbg + g^3 + 2bgg}{a}$,) let the hyperbola KM be described, with parameter $KO = \frac{f}{aa}\sqrt{aacc + 4aagd - abbg - ag^3 - 2abgg}$; in which, if it be $AP = x$, $PM = y$, it will be the *locus* of the fifth equation. From the points M, in which it cuts the circle, drawing to AP the perpendiculars MP, the lines AP, AP, will be the roots of the equation $x^4 + \frac{bff}{a}x^2 - \frac{cf^3}{a}x + \frac{df^4}{a} = 0$. And, because it was made $z = \frac{ax}{f}$, and x is given, and also z, they will be the roots of the first proposed equation.

Fig. 103.



But, if we had supposed $cc + 4gd$ to be less than $\frac{bbg + 2bgg + g^3}{a}$, the *locus* of the fifth equation would be the hyperbola MM, half the transverse axis of which $= \frac{f}{2a}\sqrt{\frac{bbg + 2bgg + g^3 - acc - 4agd}{g}}$, the conjugate semiaxis $IK = \frac{f}{2g}\sqrt{\frac{b^2g + 2bg^2 + g^3 - ac^2 - 4agd}{a}}$, and the parameter KO of the conjugate axis $= \frac{f}{a}\sqrt{\frac{bbg + 2bgg + g^3 - acc - 4agd}{a}}$.

This supposed, to satisfy the first condition, that it shall be a given circle, let it's radius be $= r$, and then it would be $r = \frac{f}{2a}\sqrt{cc - 4ad + bb - 2ab + aa}$, from which equation the value of the assumed indeterminate may be derived, or $f = \frac{2ar}{\sqrt{cc - 4ad + bb - 2ab + aa}}$. And then the circle described, EGM, will be that, the radius of which is $= r$.

To ſatisfy the ſecond condition, that the hyperbola may be given alſo, let $2t$ be the given tranſverſe axis, and p the given parameter. Then it will be $2t = \frac{f}{g}\sqrt{cc + 4gd - \frac{bbg + g^3 + 2bgg}{a}}$, and $f = \frac{2gt}{\sqrt{cc + 4gd - \frac{bbg + g^3 + 2bgg}{a}}}$.

But it is alſo $p = \frac{f}{a}\sqrt{cc + 4dg - \frac{bbg + g^3 + 2bgg}{a}}$; therefore, inſtead of f, putting it's value now found, it will be $p = \frac{2gt}{a}$, from whence we have the value of $g = \frac{ap}{2t}$. And putting this inſtead of g in the value of f, it will be $f = \frac{2apt}{\sqrt{4ttcc + 8aptd - 2bbpt - \frac{aap^3}{2t} - 2abpp}}$. Wherefore the tranſverſe diameter and the parameter of the hyperbola deſcribed (Fig. 102.) ſhall be truly the given lines $2t$ and p; and thus as to the firſt caſe.

Then, as to the ſecond, which is when $cc + 4dg$ is leſs than $\frac{bbg + g^3 + 2bgg}{a}$, let the conjugate axis of the given hyperbola be $LK = 2u$, and it's parameter $= q$; then it will be $2u = \frac{f}{g}\sqrt{\frac{bbg + 2bgg + g^3}{a} - cc - 4dg}$, and $q = \frac{f}{a}\sqrt{\frac{bbg + 2bgg + g^3}{a} - cc - 4dg}$. Whence it will be found, by operating as before, $f = \frac{2aqu}{\sqrt{2bbuq + 2baqq + \frac{aaq^3}{2u} - 4ccuu - 8aduq}}$, and $g = \frac{aq}{2u}$. And the hyperbola will have for it's conjugate axis $LK = 2u$, and for it's parameter to the ſaid axis $KO = q$. And thus the Problem will be conſtructed by means of a given circle and a given hyperbola.

Now, if the hyperbola ſhall not be given, but ought to be ſimilar to one given; that is, if the axis be to it's parameter in a given ratio, or as m to n; becauſe it has been ſeen above, that the ratio of the axis to the parameter is that of a to g, it will be ſufficient to make the analogy, $a \,.\, g :: m \,.\, n$, and thence to have the value of $g = \frac{an}{m}$.

By making uſe of the ſame method, we may conſtruct equations by means of any other given *loci*, or which are ſimilar to thoſe given. As, for example, by means of the aforeſaid given circle, and of a given ellipſis, or like to a given one, by taking the fourth equation before, inſtead of the fifth.

EX-

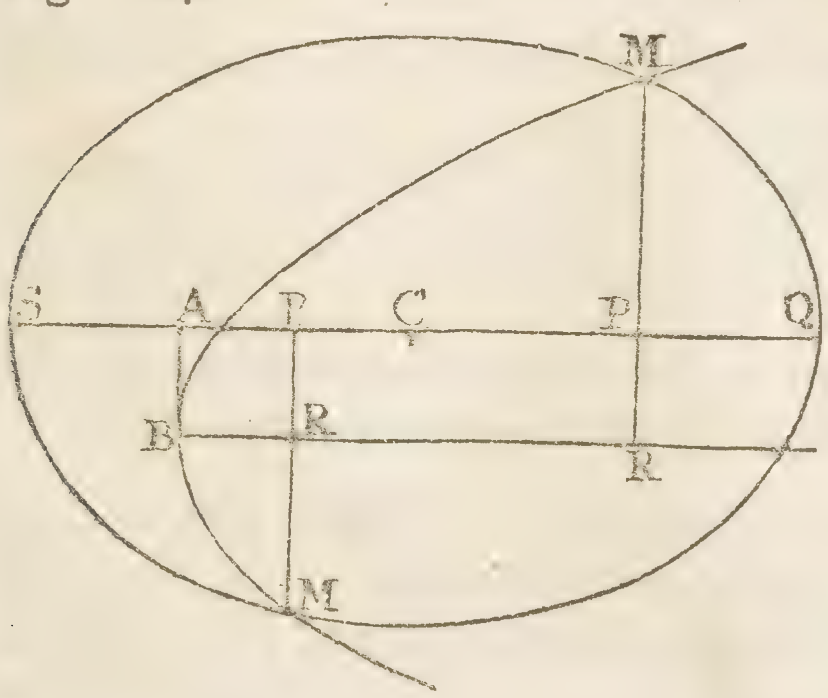
EXAMPLE V.

Let the equation be $x^4 - ax^3 - aax^2 - a^3x - 2a^4 = 0$, which it is required to construct by means of a parabola whose parameter $= a$, and of an ellipsis similar to one given, whose transverse axis is to the parameter in the given ratio of b to d.

Let the second term be taken away by the substitution of $x = z + \frac{1}{4}a$, and the transformed equation will be $z^4 * - \frac{11}{8}aaz^2 - \frac{13}{8}a^3z - \frac{595}{256}a^4 = 0$. I put $z = \frac{ay}{f}$, to introduce the first indeterminate f, and it will be $y^4 - \frac{11}{8}ffy^2 - \frac{13}{8}f^3y - \frac{595}{256}f^4 = 0$. Now, taking for the first *locus* $yy = fq$ to the parabola, and making a substitution of the values of y^4 and y^2, we shall have the second *locus* also to the parabola, $qq - \frac{11}{8}fq - \frac{13}{8}fy - \frac{595}{256}ff = 0$. Now, because the given parabola has it's parameter $= a$, we may here make use of the first *locus*, by taking $f = a$, and therefore it will be $yy = aq$. And substituting the value of f in the second, (for, the ellipsis not being given, the first indeterminate f, in respect of this, is still arbitrary,) it will be $qq - \frac{11}{8}aq - \frac{13}{8}ay - \frac{595}{256}aa = 0$.

Now let the first *locus* be multiplied by $\frac{g}{a}$, in order to introduce the second indeterminate g, and it will be $\frac{gyy - agq}{a} = 0$, which, being added to the second, will give the third *locus*, $qq - \frac{11}{8}aq - \frac{13}{8}ay - \frac{595}{256}a^2 + \frac{gyy - agq}{a} = 0$, which is to an ellipsis.

Fig. 104.



For the construction of this third *locus*, we should have the ellipsis MSQ to describe, with the transverse axis SQ $= 2\sqrt{\frac{716a^2g + 176ag^2 + 64g^3 + 169a^3}{256g}}$, and wit parameter $= \frac{2a}{g}\sqrt{\frac{716a^2g + 176ag^2 + 64g^3 + 169a^3}{256g}}$. But, because in this the ratio of the axis to the parameter is that of g to a, which, by the given condition, ought to be that of b to d, it will be $g = \frac{ab}{d}$. And therefore, instead of g, substituting it's value,

value, the ellipsis MSQ must be described with the transverse axis $= \frac{1}{8d}\sqrt{\frac{716a^2bd^2 + 176a^2b^2d + 64a^2b^3 + 169a^2d^3}{b}}$, and with parameter $= \frac{1}{8b}\sqrt{\frac{716a^2bd^2 + 176a^2b^2d + 64a^2b^3 + 169a^2d^3}{b}}$.

Now, from the centre C taking CA $= \frac{11ad + 8ab}{16d}$, and from the point A letting fall the perpendicular AB $= \frac{13d}{16b}$, if from the point B be drawn BR parallel to the axis SQ, taking any line BR $= q$, it will be RM $= y$, and the ellipsis will be the *locus* of the third equation $qq - \frac{11}{8}aq - \frac{13}{8}ay - \frac{595}{256}aa + \frac{gyy - agq}{a} = 0$.

With vertex B, axis BR, and parameter $= a$, let the parabola MBM of the equation $yy = aq$ be described; it will cut the ellipsis in two points M, M. From which points drawing RM, RM, perpendicular to the right line BR, they will be the two real roots of the proposed equation.

For, by the property of the ellipsis, it will be SP $\times$ PQ to PMq, so is the transverse axis to the parameter. But CP $= q - \frac{11ad + 8ab}{16d}$, and therefore SP $= \frac{1}{16d}\sqrt{\frac{716a^2bd^2 + 176a^2db^2 + 64a^2b^3 + 169a^2d^3}{b}} + q - \frac{11ad + 8ab}{16d}$, and PQ $= \frac{1}{16d}\sqrt{\frac{716a^2bd^2 + 176a^2db^2 + 64a^2b^3 + 169a^2d^3}{b}} - q + \frac{11ad + 8ab}{16d}$. And besides, PM $= y - \frac{13ad}{16b}$. Therefore we shall have the analogy, $\frac{716a^2bd^2 + 176a^2b^2d + 64a^2b^3 + 169a^2d^3}{256bd^2} - q^2 + \frac{11daq + 8baq}{8d} - \frac{121a^2d^2 + 176a^2bd + 64a^2b^2}{256d^2}$. $yy - \frac{13ady}{8b} + \frac{169a^2d^2}{256b^2} :: \frac{1}{d} . \frac{1}{b} :: b . d$. And therefore the equation $\frac{595a^2bd^2}{256bd} - dqq + \frac{11adq + 8abq}{8} = byy - \frac{13ad}{8}y$. But, by the equation to the parabola, it is $yy = aq$. Therefore, substituting, instead of q and qq, their values $\frac{yy}{a}$ and $\frac{y^4}{a^2}$, and ordering the equation, dividing by d and multiplying the terms by aa, it will be $y^4 - \frac{11aayy}{8} - \frac{13a^3y}{8} - \frac{595a^4}{256} = 0$. But, by making the substitution of $z = \frac{ay}{f}$, (or making $y = z$, for $a = f$,) it will be $z^4 - \frac{11}{8}a^2z^2 - \frac{13}{8}a^3z - \frac{595}{256}a^4 = 0$, which is the reduced equation; to the roots of which adding $\frac{1}{4}a$, they will be the roots of the equation proposed.

It was indeed unnecessary to take all this trouble about an Example, which, by nature, is not solid but plane; for the proposed equation is divisible by $x + a$, and by $x - 2a$. But, however, it will serve to show the use of this method.

202. Equa-

202. Equations of the fifth and fixth degree are conftructed by means of two *loci*, one of the third degree, and the other a conic fection.

Equations conftructed of the fifth and fixth degree.

EXAMPLE VI.

Let the equation be $x^5 + aax^3 - a^5 = 0$. I take the *Apollonian* parabola $xx = ay$, and making the fubftitutions, there arifes the fecond *locus* $xyy + axy - a^3 = 0$.

Hitherto I have not mentioned the conftruction of *loci* above the Conic Sections, having referved the treating on thefe for the following Section; for thus order neceffarily required. At prefent, therefore, let there be fuppofed, and alfo let there be defcribed, a curve with three branches MCH, FE, PNO, whofe equation is $xyy + axy - a^3 = 0$, in which AB reprefents the x's, and BC the y's. With vertex A, axis AL, and parameter $= a$, let the *Apollonian* parabola RAC be defcribed. It will meet the branch MCH in the point C; and therefore, letting fall the perpendicular CB, it will be AB $= x$, the real and pofitive root of the propofed equation, and the other four will be imaginary. If we defire to conftruct the fame equation by means of an hyperbola between it's afymptotes, and alfo by a *locus* of the third degree, make $xy = aa$, and, by fubftituting, it will be $x^3 + aax - ayy = 0$.

Fig. 105.

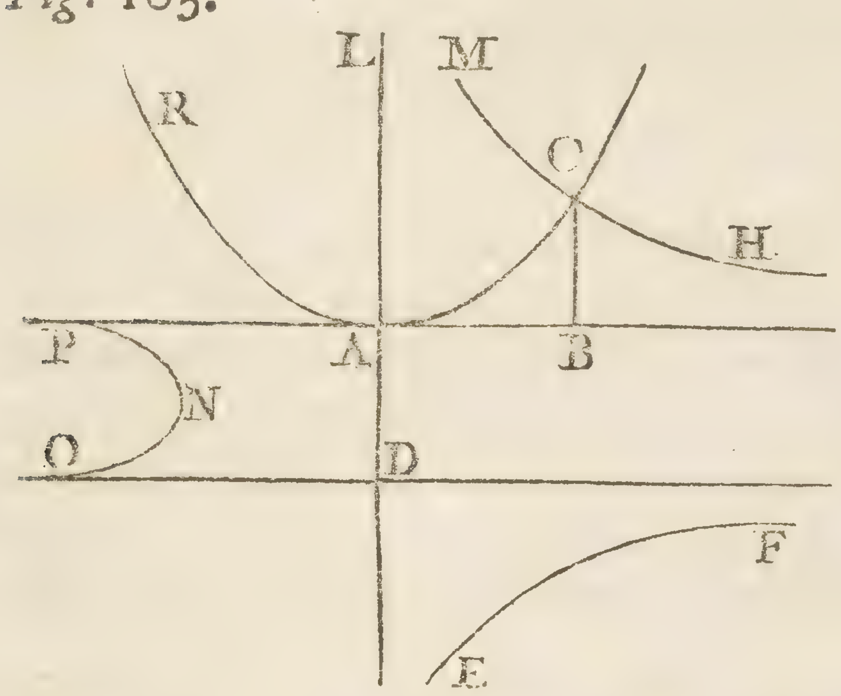
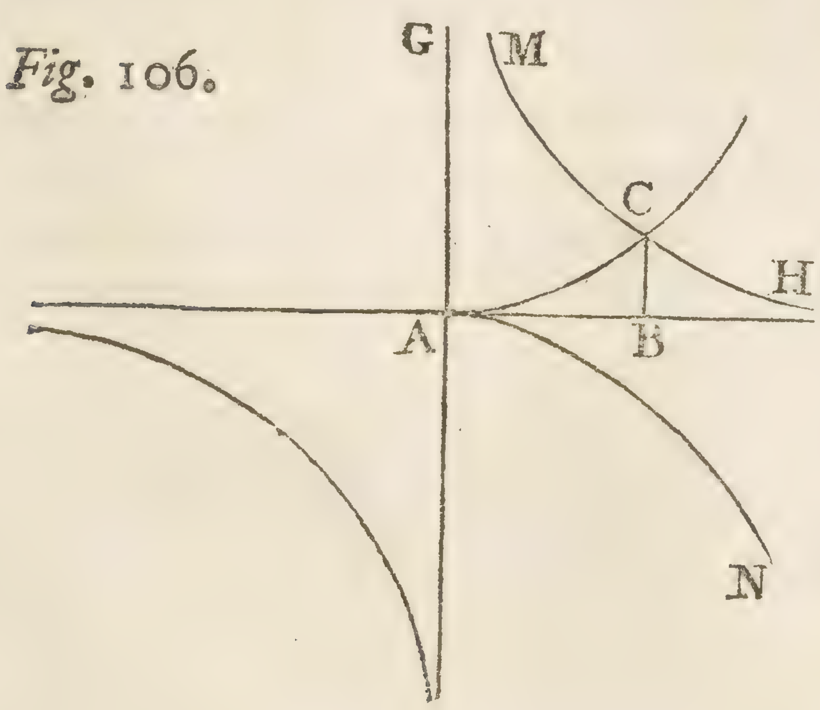


Fig. 106.

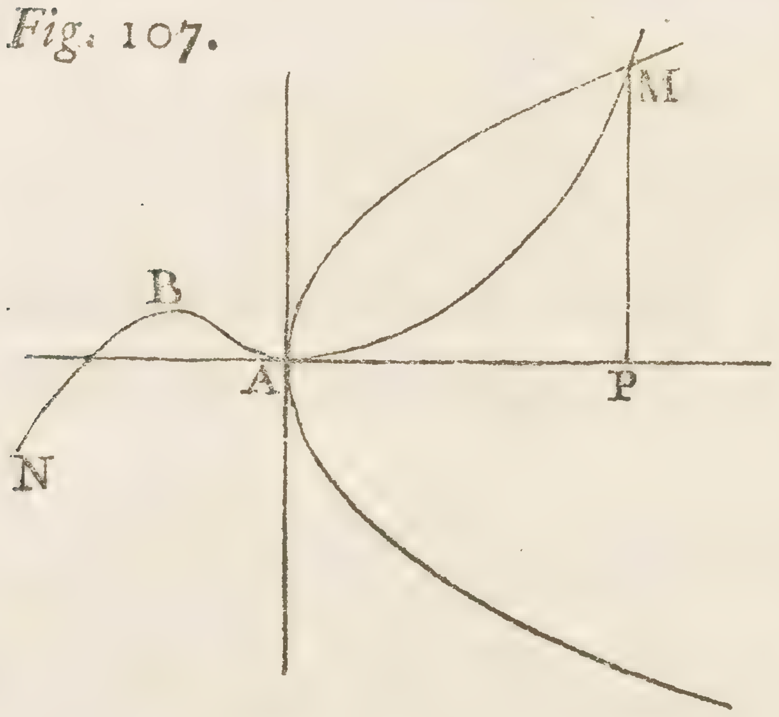


To axis AB, with abfcifs AB $= x$, and ordinate BC $= y$, let the curve CAN be defcribed, which is the *locus* of the equation $x^3 + aax - ayy = 0$. And between the afymptotes AB, AG, let the hyperbola MCH of the equation $xy = aa$ be defcribed, taking the x's on the fame axis AB; this will cut the firft curve in the point C, from whence letting fall the perpendicular CB, it will be AB $= x$, the root of the equation propofed.

Now

Now I multiply the ſame equation by $x = 0$, in order to reduce it to the ſixth degree, and I ſhall have $x^6 + aax^4 - a^5x = 0$. I take the ſame *locus* to the parabola $xx = ay$, and, making the ſubſtitution, there ariſes the ſecond *locus* $y^3 + ay^2 - aax = 0$, which is the curve NBAM, taking the abſciſſæ AP $= y$, and the ordinates PM $= x$.

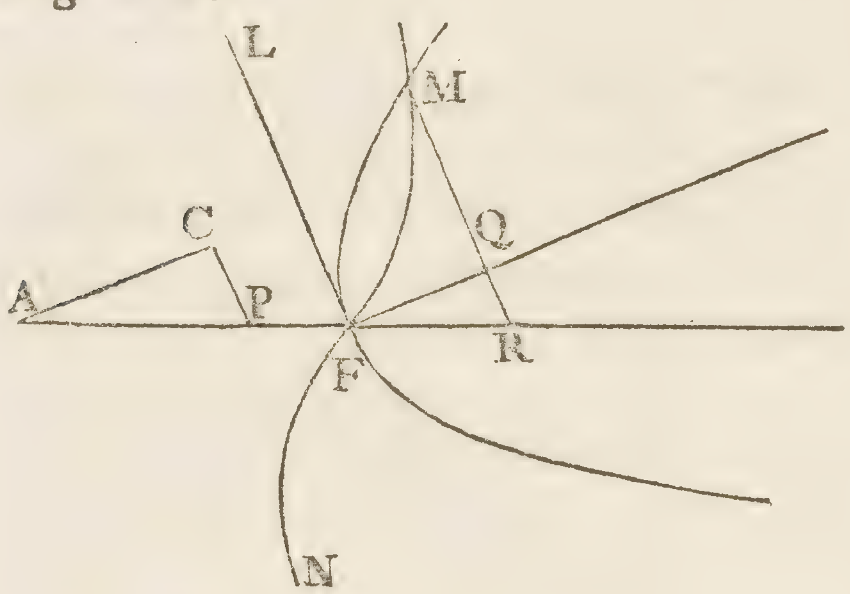
Fig. 107.



With vertex A, to the axis AP, with parameter $= a$, the *Apollonian* parabola AM of the equation $xx = ay$ being deſcribed, it will cut the ſaid curve in the vertex A, which gives us one of the roots $x = 0$, the ſame that was introduced into the equation. Beſides, it will cut it in the point M, and letting fall the perpendicular MP, it will be another root of the equation.

If we deſire to make uſe of the firſt cubic parabola $x^3 = aay$, make the ſubſtitution in the equation $x^6 + a^2x^4 - a^5x = 0$, and there ariſes the ſecond *locus*, $yy + xy - ax = 0$, to the *Apollonian* hyperbola.

Fig. 108.



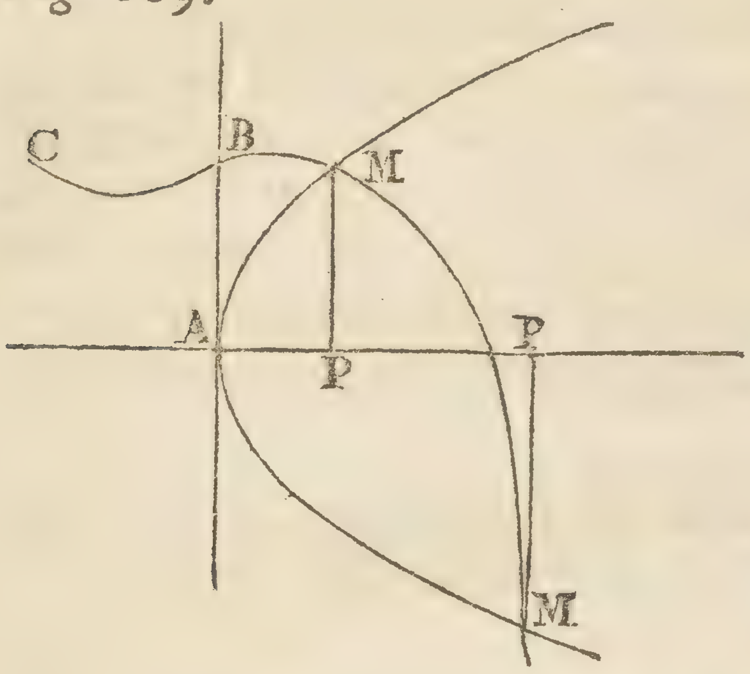
On the indefinite line AP let the triangle ACP be deſcribed, being right-angled at C, (ſuppoſing, if you pleaſe, the angle of the co-ordinates of the equation $yy + xy - ax = 0$ to be right,) and let it be AC . CP :: 2 . 1. At the centre A, with the tranſverſe ſemidiameter AF $= a\sqrt{5}$, with the parameter $= \frac{2a}{\sqrt{5}}$, let the *Apollonian* hyperbola FM be deſcribed; then from the point F drawing the indefinite line FQ parallel to AC, and taking any line FQ $= x$, and QM parallel to CP and equal to y, this ſhall be the *locus* of the equation $yy + xy - ax = 0$. To the axis FL parallel to PC, let there be deſcribed the cubical parabola NFM of the equation $x^3 = aay$. This will cut the hyperbola in the vertex F, which gives us the root $x = 0$. And from the point M letting fall the perpendicular MQ upon FQ, this will determine the other root FQ of the equation $x^6 + aax^4 - a^5x$.

If our equation had had the ſecond term, and if we had deſired to make uſe of the cubic parabola, a ſecond *locus* of the third degree would have been derived. Therefore we ought to make the ſecond term to vaniſh, or make uſe of another *locus*.

EXAMPLE VII.

Let the equation of the fixth degree be this, $x^6 + ax^5 + a^5x - a^6 = 0$. I take the *locus* to the *Apollonian* parabola $xx = ay$. Making the fubftitutions, the fecond *locus* will be $y^3 + xy^2 + aax - a^3 = 0$, which is the curve CBM, taking the abfciffes AP $= y$, and the ordinates PM $= x$.

Fig. 109.

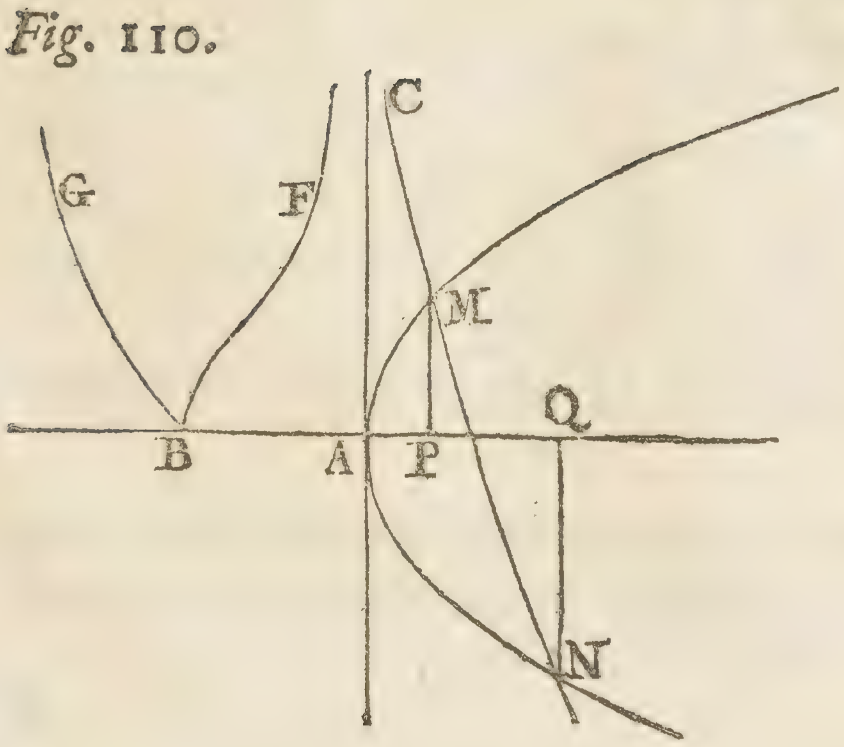


At the vertex A, with parameter $= a$, to the axis AP, let the parabola MAM of the equation $xx = ay$ be defcribed. This will cut the faid curve in two points M, M, from whence drawing to the axis the perpendiculars MP, MP, they will be the two roots of the propofed equation, of which one will be pofitive, the other negative, and the four others will be imaginary.

203. Equations of the feventh degree are conftructed by means of two *loci* of the third, or elfe by one of the fecond and one of the fourth. But, becaufe, by multiplying them by the unknown quantity, they are reduced to the eighth degree, and thofe of the eighth are conftructed in like manner by a *locus* of the fecond, and another of the fourth, I fhall content myfelf with giving an Example of thofe of the eighth degree. —of the feventh or eighth degree.

EXAMPLE VIII.

Fig. 110.



Let the equation of the eighth degree be $x^8 + ax^7 + a^3x^5 - a^8 = 0$. Taking the equation to the *Apollonian* parabola $xx = ay$, and making the fubftitutions, there arifes the fecond *locus* $x^4 + xy^3 + axy^2 - a^4 = 0$, which is the curve GBFCMN, taking the abfciffes AP $= y$, and the ordinates PM $= x$. With vertex A, parameter $= a$, and axis AP, let the parabola of *Apollonius*, MAN, be defcribed, belonging to the equation $xx = ay$.

 This

This will meet the aforesaid curve in the points M, N, from which drawing the perpendiculars MP, NQ, to the axis, they will be the two real roots, one positive, the other negative, of the proposed equation, and the others are imaginary.

—or of higher degrees.

204. Here it may be observed, that equations of the ninth degree, (as well as those of the eighth, reduced to the ninth by multiplying them by the unknown quantity,) may always be constructed by means of two *loci* of the third degree, making the second term to vanish, if it have one.

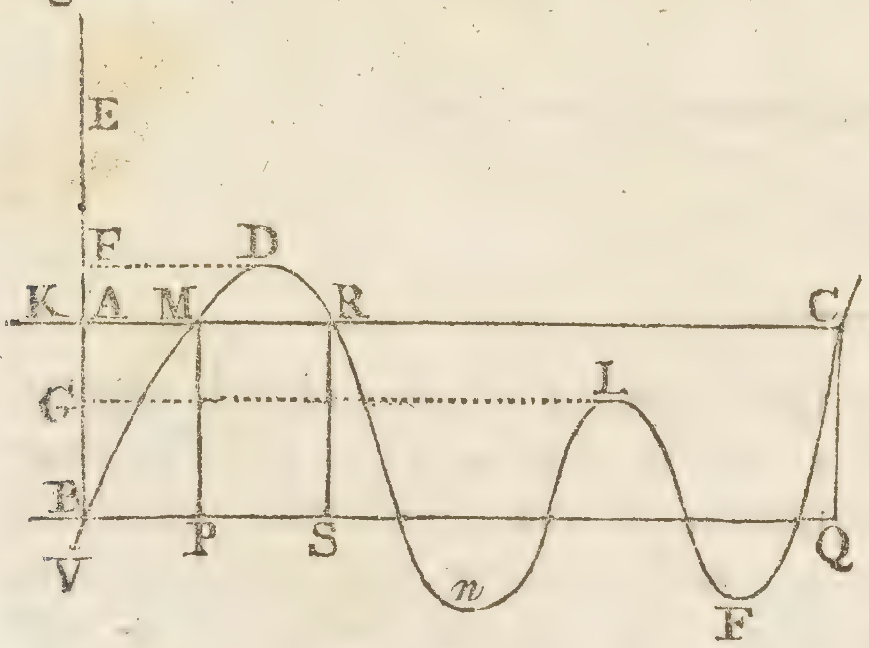
Thus, in general, equations of the tenth degree may be constructed by means of a *locus* of the third degree, and one of the fourth. And, in like manner, those of eleven and twelve degrees, observing to reduce those of eleven to twelve, by multiplying them by the unknown quantity, and by making the second term of an equation of the twelfth degree to vanish, if it have any. And the like is to be understood of equations of higher degrees.

All equations may be constructed by a *locus* of the same degree.

205. Another manner of constructing equations of any degree may be, by means of a *locus* of the same degree as the equation proposed, and a right line; after the following manner.

Let it be an equation of the fifth degree, $x^5 - bx^4 + acx^3 - aadx^2 + a^3cx - a^4f = 0$. Let the last term a^4f be transposed, and taking one of the linear divisors, f, of the last term, make it equal to z, for example, and divide the equation by a^4; then we shall have $z = \frac{x^5 - bx^4 + acx^3 - a^2dx^2 + a^3cx}{a^4}$.

Fig. 111.



On the indefinite line BQ describe the curve BMDRNLFC of this last equation, taking the x's from the fixed point B. The ordinates PM, SR, &c. will be equal to z; and therefore, from the point B draw the right line BA $= f$, parallel to the ordinates PM, SR, and through the point A draw the indefinite right line KC both ways, and parallel to BQ. From the points in which it cuts the curve, let fall the perpendiculars MP, RS, CQ; they will determine the abscisses BP, BS, BQ, which are the roots of the equation proposed. Those from A towards Q are positive, and those the contrary way are negative.

If the right line AC shall touch the curve in any point, the corresponding abscis x shall denote two equal roots; and if it meet it in no point, all the roots will be imaginary.

If the laſt term had had it's ſign poſitive, we muſt have made $z = -f$, and therefore muſt have taken BA $= -f$, that is, below the point B, or on the negative ſide.

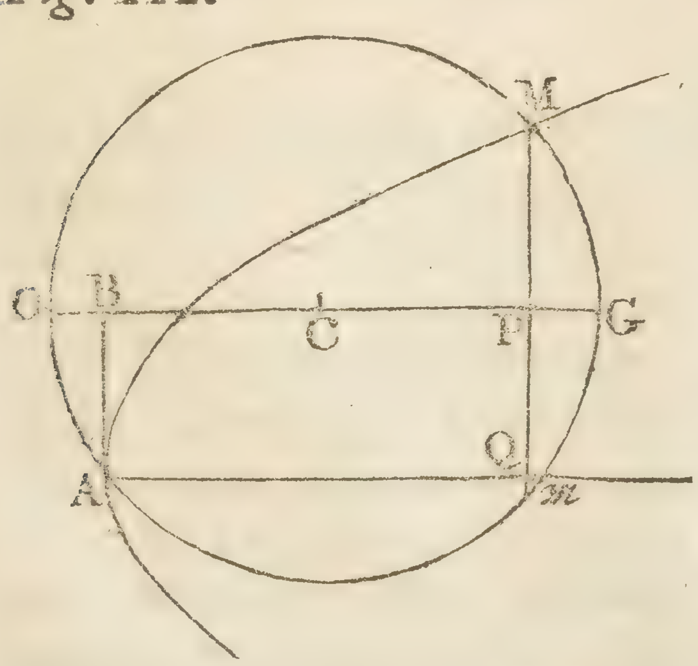
206. This method may be of uſe to verify conſtructions, which have been made by the combination of two curves, by confronting with each other the number of the roots, whether real or imaginary, poſitive or negative, which are found by each method. Uſe of this method.

PROBLEM I.

207. Between two given quantities, to find as many mean geometrical proportionals as ſhall be required. A Problem to exemplify this method.

Let the two given quantities be a and b, and let x be the firſt of the mean proportionals; they will form this geometrical progreſſion following: a, x, $\frac{x^2}{a}$, $\frac{x^3}{a^2}$, $\frac{x^4}{a^3}$, $\frac{x^5}{a^4}$, &c. Now, if we would have two mean proportionals, the fourth term of the progreſſion muſt be b, and therefore we ſhould have this equation $\frac{x^3}{a^2} = b$, or $x^3 = a^2b$. To conſtruct this by the help of a parabola and a circle, I reduce it to the fourth degree, by multiplying it by $x = 0$, and then it will be $x^4 - a^2bx = 0$. Taking the *locus* to the parabola $xx = ay$, and making the ſubſtitutions, there ariſes the ſecond *locus* $yy - bx = 0$, which is alſo to the parabola; from which ſubtracting the firſt, there ariſes a third, $yy - bx - xx + ay = 0$, which is to the hyperbola; or, adding the firſt and ſecond together, there ariſes, laſtly, $yy - bx + xx - ay = 0$, a *locus* to the circle, ſuppoſing the co-ordinates to contain a right angle.

Fig. 112.



With radius CG $= \frac{1}{2}\sqrt{aa + bb}$ let the circle OMA be deſcribed; and taking CB $= \frac{1}{2}a$, let fall the perpendicular BA $= \frac{1}{2}b$, which will meet the circle in the point A; from whence drawing AQ parallel to the diameter OG, and taking any portion AQ $= y$, it will be QM $= x$, and this circle will be the *locus* of the equation $yy - bx + xx - ay = 0$. With vertex A, axis AQ, and parameter $= a$, let the parabola $xx = ay$ be deſcribed, which will meet the circle in the point M; from whence letting fall the perpendicular MQ, it will be the root of the propoſed equation. For the vertex of the pa-

 rabola,

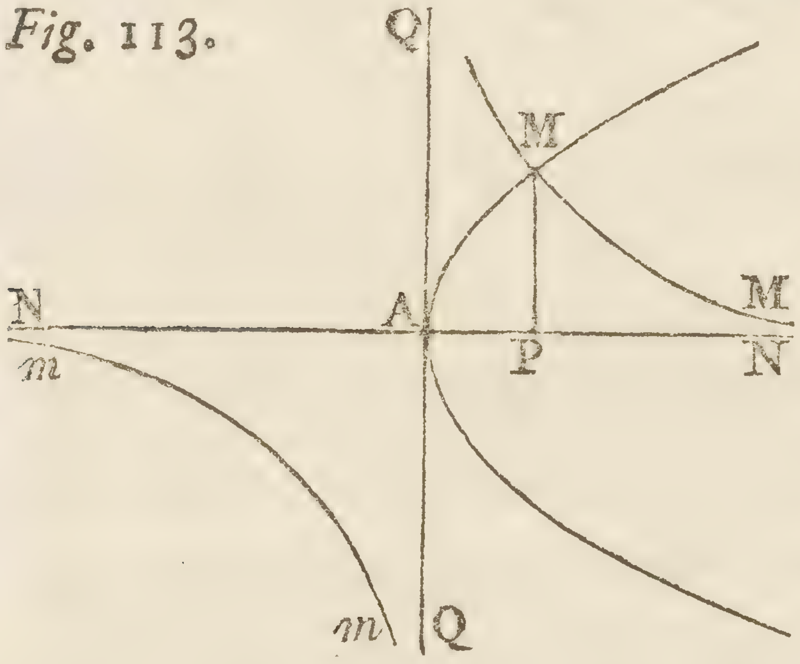
rabola, being in the periphery of the circle, will give the other root $x = 0$, which was introduced, and the other two are imaginary.

Taking the firft and fecond equation, the Problem will be conftructed by means of two *Apollonian* parabolas. Taking the firft and third, it will be conftructed by means of the parabola, and the hyperbola referred to it's diameters.

The fame otherwife conftructed.

208. Without multiplying the equation $x^3 - aab = 0$, it may be conftructed by a parabola and an hyperbola between it's afymptotes; for, taking the *locus* $xx = ay$, and making the fubftitutions, there arifes $xy = ab$.

Fig. 113.



Between the afymptotes NN, QQ, let there be defcribed the hyperbola MM with the conftant rectangle ab, and let AP be the y's, and PM the x's. To the axis AP, with the vertex A, the parameter $= a$, let the parabola AM be defcribed; from the point M, in which it cuts the hyperbola, drawing the ordinate MP, it fhall be the root of the propofed equation.

The firft of the two mean proportionals being thus found, we have alfo the fecond, being equal to the abfcifs AP $= y = \frac{xx}{a}$.

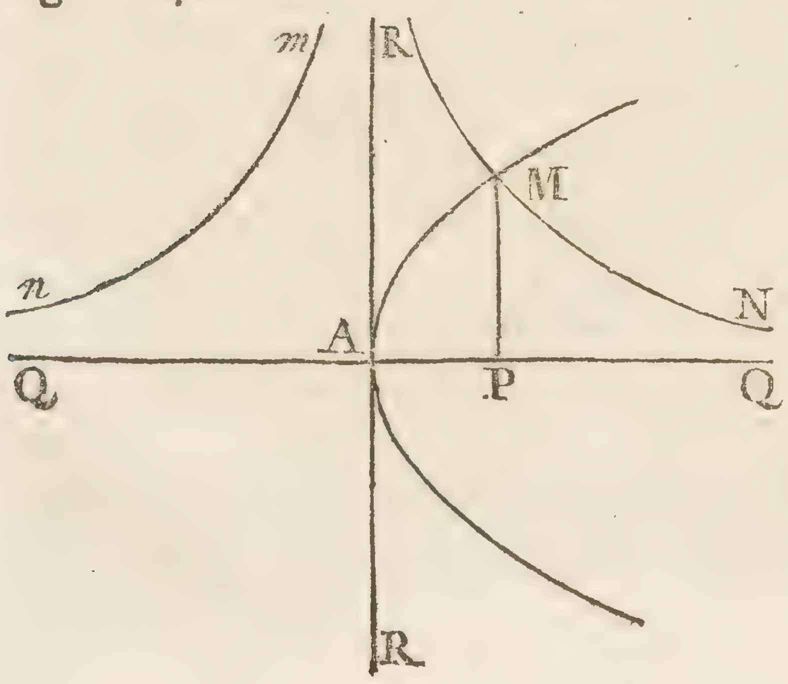
A fimpler cafe of the fame Problem.

209. To find three mean proportionals, the Problem becomes plane; for, having found, geometrically, that in the middle, which let be m for example, the mean between a and m will be the firft of the three, and the mean between m and b will be the third.

Carried higher.

210. Let it be required to find four mean proportionals; then b ought to be the fixth term of the progreffion, and therefore we fhall have the equation $x^5 = a^4b$, or $x^5 - a^4b = 0$.

Fig. 114.



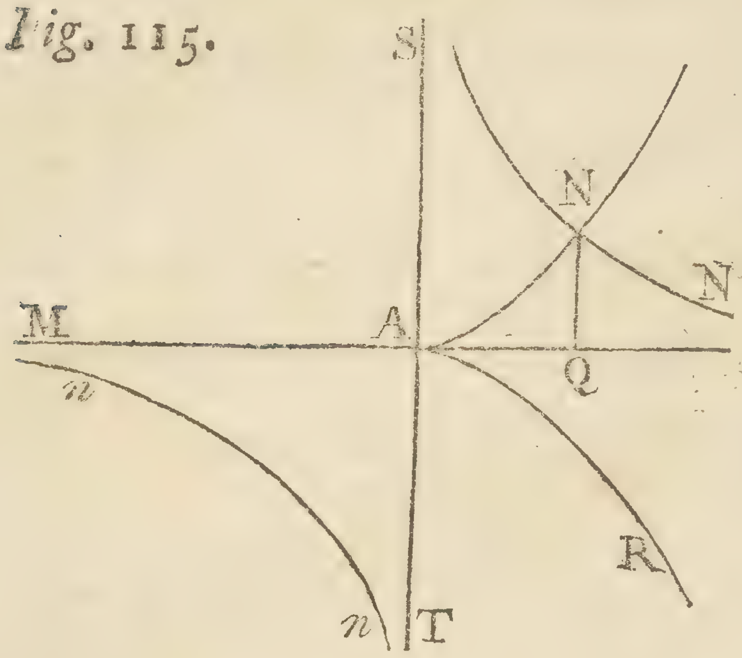
I take the *locus* to the *Apollonian* parabola $xx = ay$, and making the fubftitution, there arifes the fecond *locus* $xyy - aab = 0$, which is an hyperboloid of the third degree. Therefore, between the afymptotes QQ, RR, let there be defcribed the hyperboloid MN, mn, of the equation $xyy = aab$, making the abfcifs AP $= y$, and the ordinate PM $= x$. Now, to the diameter AQ, vertex A, defcribing the parabola of the equation $xx = ay$; and from the point M, in which it meets the

the hyperboloid, drawing the ordinate MP, it ſhall be the root of the equation $x^5 - a^4b = 0$, and the firſt of the mean proportionals required; by means of which the others may be found alſo.

211. Alſo, the Problem may be conſtructed by means of the *Apollonian* hyperbola between it's aſymptotes, and the ſecond cubical parabola. Conſtructed otherwiſe.

Make therefore $aa = xy$, the *locus* to the aforeſaid hyperbola; and, inſtead of a^4, ſubſtituting it's value $xxyy$, there ariſes the *locus* $x^3 = byy$, which is the ſecond cubical parabola.

Fig. 115.



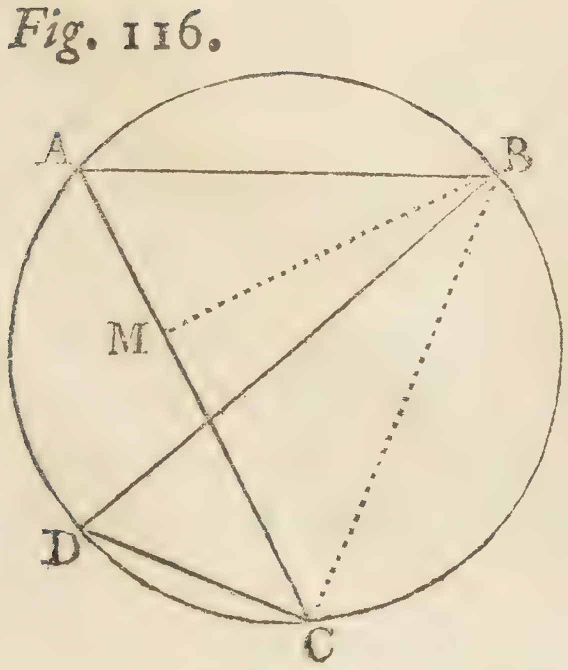
With the axis AQ let there be deſcribed the ſecond cubic parabola RAN, in which AQ gives the x's, and QN the y's. And between the aſymptotes ST, MQ, let there be deſcribed the hyperbola NN. And from the point N, in which it meets the parabola, let the ordinate NQ be drawn. Then will AQ be the root of the propoſed equation, that is, the firſt of the four mean proportionals.

212. To find five mean proportionals the Problem is only cubical. For, having found the middle term geometrically, which, for example, let be m; to have the two means between a and m, is a cubical or ſolid Problem, as has been ſeen juſt now. Extended to higher caſes.

It may be eaſily perceived with a little attention, that the Problem for finding ſix mean proportionals may be conſtructed, either with a *locus* of the ſecond, and one of the fourth degree, or with two of the third degree. But to find ſeven ſuch, having found the middle one, the Problem will be reduced to the finding of three. And in the ſame way of reaſoning, we may go on to greater numbers.

PROBLEM II.

Fig. 116.



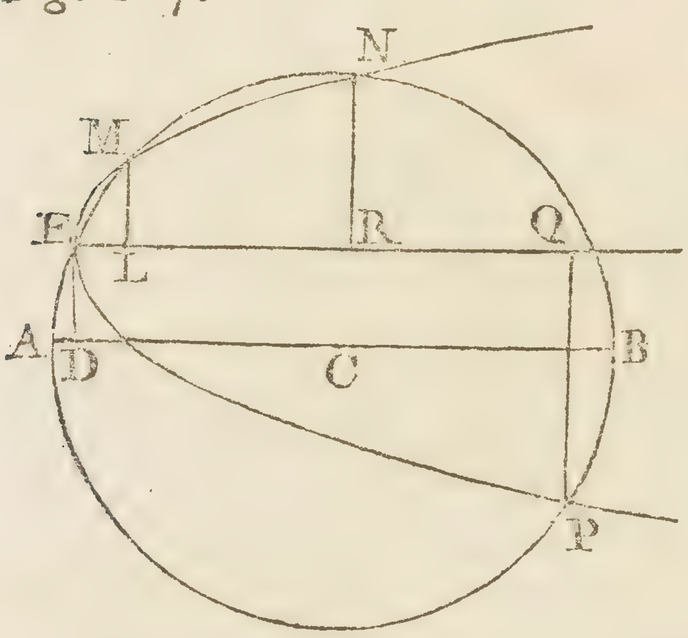
213. In the circle ABCD, having two chords given, BA, DC, which proceed from the extremities of the diameter BD, and the third chord AC being given alſo; to find the diameter BD. The *loci* exemplified by another Problem.

Draw the chord BC, and make $AB = a$, $AC = b$, $DC = c$, and the diameter $BD = x$; and let fall the perpendicular BM upon the chord AC. Becauſe the angle in the ſemicircle BCD is a right one, it will be $BC = \sqrt{xx - cc}$; and becauſe the angles BAC, BDC, inſiſt

insist on the same arch BC, and also the angles M, BCD, are right angles, the two triangles BCD, BAM, will be similar. Wherefore it will be AM $= \frac{ac}{x}$. But, by *Euclid*, ii. 13, it is BCq = ABq + ACq — 2CA $\times$ AM; therefore the equation will be $xx - cc = aa + bb - \frac{2abc}{x}$, that is, $x^3 - ccx - aax - bbx + 2abc = 0$.

I multiply it by x, to reduce it to the fourth degree, and thus construct it, by means of the parabola and the circle. It is then $x^4 - c^2x^2 - a^2x^2 - b^2x^2 + 2abcx = 0$. Taking therefore the *locus* to the parabola, the parameter of which is the least of the three chords, which let be c for instance; that is, taking $xx = cy$, make the substitutions, and the second *locus* will arise $yy - \frac{ccy + aay + bby}{c} + \frac{2abx}{c} = 0$, which is also to the parabola. To this add the first equation $xx - cy = 0$, and we shall have finally a *locus* to a circle, taking the co-ordinates at right angles, that is, $yy - \frac{2cc + aa + bb}{c}y + \frac{2ab}{c}x + xx = 0$.

Fig. 117.

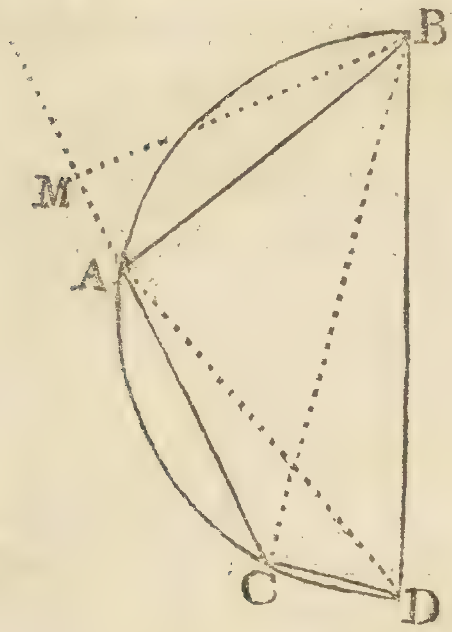


Therefore, with radius AC $= \sqrt{\frac{aabb + ccmm}{cc}}$, (for brevity-sake writing m for $\frac{2cc + aa + bb}{2c}$,) draw the circle AMBP, and taking CD $= m$, from the point D raise the perpendicular DE $= \frac{ab}{c}$, which will terminate in the periphery of the circle at the point E; and drawing the indefinite line EQ parallel to the diameter AB, upon this line take any how EL $= y$, the corresponding ordinate will be LM $= x$, and this circle is the *locus* of the equation. With vertex E, axis EQ, and parameter $= c$, let the parabola of the equation $xx = cy$ be described. This will cut the circle at the vertex in the point E, which will give the introduced root $x = 0$. It will cut it besides in the three points M, N, P, from whence, to the right line EQ letting fall the perpendiculars ML, NR, PQ, they shall be the three roots of the equation proposed, two positive and one negative. The first positive root ML cannot serve for this Problem; for, supposing $y = c$, it will be in the parabola, $x = c$, and in the circle, $x = -\frac{ab}{c} + \sqrt{\frac{aabb}{cc} + bb + aa + cc}$. But this value of x, relatively to the circle, is greater than c, if the two chords a, b, be not equal to each other; and it is equal to c, if the two chords be equal. Wherefore the point in the parabola which corresponds to the abscis c, either falls in M, or falls within the circle. Therefore ML is either less than c, or,

or, at most, is equal to it, and therefore must needs be less than either of the chords a, b, and consequently cannot be the diameter of the circle.

The second positive root RN will supply us with the diameter required. The negative root QP supplies us with a diameter for another case; that is, when the two chords which terminate at the diameter are drawn from the same side, as in Fig. 118. For, doing the same things as above, draw likewise the chord AD. The angle DAB being right, the two angles DAC, MAB, will be equal to a right angle. But also, the two angles MAB, MBA, are equal to a right angle; therefore MBA = DAC = CBD, as insisting on the same arch DC. Hence the two triangles CBD, MBA, are similar, and therefore MA $= \frac{ac}{x}$; but, by *Euclid*, ii. 12, it will be $CBq = CAq + BAq + 2CA \times AM$; whence the equation $xx - cc = bb + aa + \frac{2abc}{x}$, that is, $x^3 - ccx - bbx - aax - 2abc = 0$; the construction of which is the same as the preceding, except that now, the last term being negative, we must draw DE (Fig. 117.) the negative way, because the axis of the parabola will be below the diameter of the circle; and the two positive roots in the first case are negative in this, and the negative becomes positive.

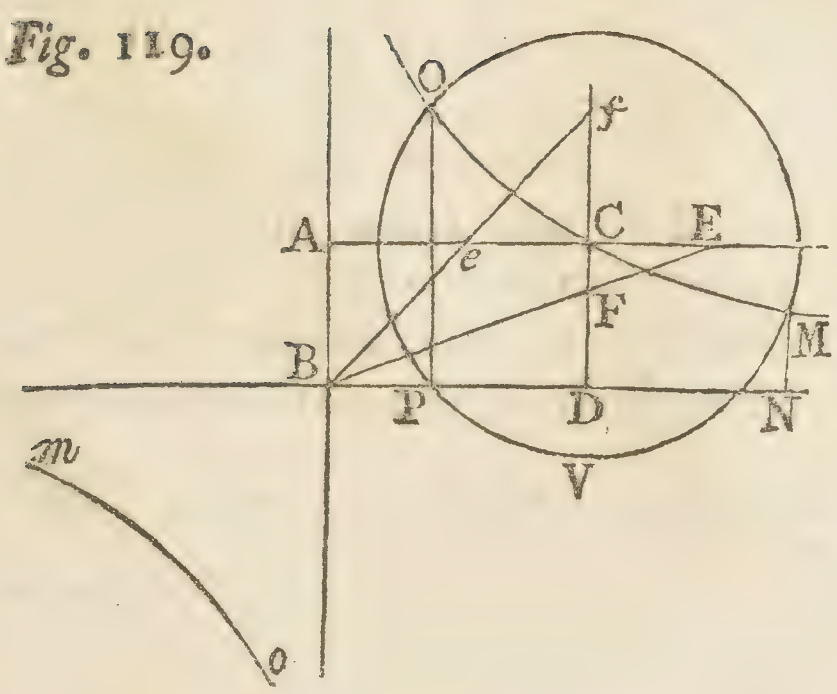
Fig. 118.



And because the second term is wanting in both the equations, it proceeds from thence, that the two positive roots in the first case are equal to the negative, and the positive in the second is equal to the two negative. Hence we learn that the first of the three roots, which gave us no solution of the Problem, yet however belonged to it, as being the difference of the two diameters.

PROBLEM III.

Fig. 119.



214. The rectangle ACDB being given, in the side AC produced to find the point E, so that, drawing the right line BE from the angle B, the intercepted line EF may be equal to a given right line c. Another geometrical Problem.

When a square is given instead of the rectangle ABDC, the Problem is plane, and has been already solved in Sect. IV. § 176. But, supposing ABDC to be a rectangle,

rectangle, it changes the nature of the Problem, and makes it solid. Therefore, making AB $= a$, BD $= b$, DF $= x$, and repeating the argumentation in the place above cited, we shall have an equation of the fourth degree, which is this:

$$\begin{array}{llll} x^4 - 2ax^3 & + aax^2 & - 2abbx & + aabb = 0. \\ & + bbx^2 & & \\ & - ccx^2 & & \end{array}$$

To construct this by an hyperbola between the asymptotes, combined with a circle, I put $ab = zx$, and making the substitutions, there arises the second *locus* $xx - 2ax + aa + bb - cc - 2bz + zz = 0$, which is to the circle.

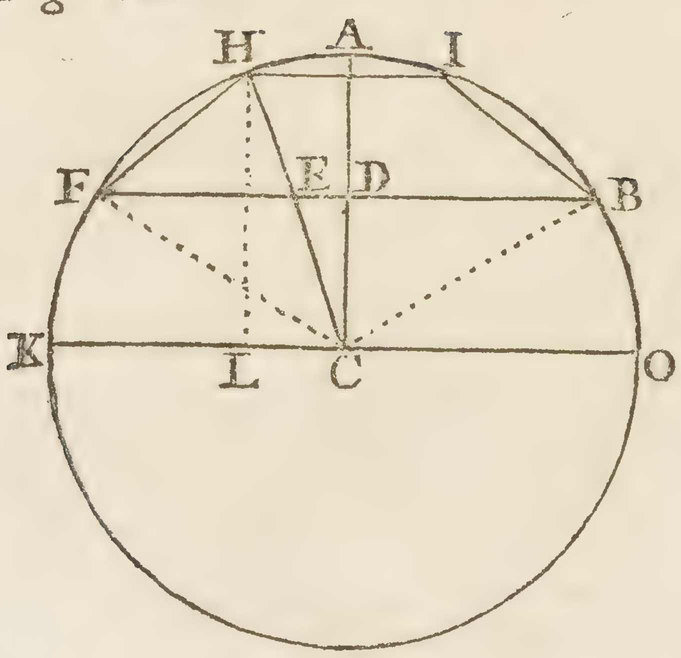
Between the asymptotes BA, BD, let the hyperbola OM be described, of the equation $zx = ab$, which shall pass through the point C. Taking any abscis BP, BN $= z$, the ordinate will be PO, NM $= x$. With centre C, radius equal to the given line c, let the circle OMV be described. This shall be the *locus* of the equation $xx - 2ax + aa + bb - cc - 2bz + zz = 0$.

From the points O, M, in which this cuts the hyperbola, let fall the perpendiculars OP, MN; they shall be the two positive roots of the equation. The lesser will serve for the Problem in the case proposed, of the angle BAC. The greater for the angle ACf. And if the given line c be such, that the circle cannot reach to cut the opposite hyperbola *mo*, the other two roots will be imaginary. But if it shall cut it, they will be real and negative, and will serve for the angle ACD.

PROBLEM IV.

A Problem for angular sections.

Fig. 120.



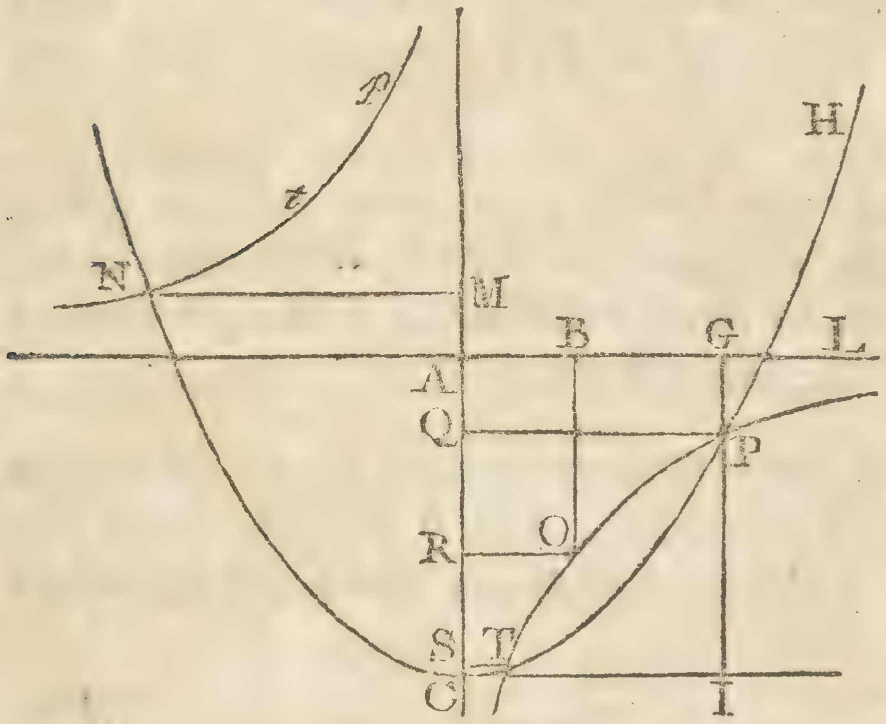
215. To divide a given angle FCB, or arch FAB, into three equal parts.

Let H, I, be the points of division required; then the chords FH, HI, IB, ought to be equal: and the arch FAB being given, it's chord FB will also be given, which let be equal to $2f$. Then, drawing the radius CA $= r$ perpendicular to FB, which will bisect it in D, it will also bisect the chord HI, and CD will be known, which make $= a$. Drawing the radius CK perpendicular to CA, and from the point H drawing HL perpendicular to CK, make CL $= y$, and it will be, by the property of the circle, HL $= \sqrt{rr - yy}$. And drawing the radius CH, by the similar triangles HLC, CDE, we shall have DE $= \frac{ay}{\sqrt{rr - yy}}$. But, because the angle FHC

 ought

ought to be equal to the angle CHI, by the conditions of the Problem, and CHI $=$ CED by the parallels FB, HI, and CED $=$ FEH; then FHC $=$ FEH, and therefore FE $=$ FH. But FH $=$ HI $= 2y$, therefore FE $= 2y$. And the whole line FD $= 2y + \frac{ay}{\sqrt{rr - yy}}$. But FD $= f$; therefore $2y + \frac{ay}{\sqrt{rr - yy}} = f$; and taking away the aſymmetry, it will be $y^4 - fy^3 + \frac{1}{4}ffyy + \frac{1}{4}aayy - rryy + frry - \frac{1}{4}ffrr = 0$; or, becauſe $rr = ff + aa$, it is $y^4 - fy^3 - \frac{3}{4}rry^2 + frry - \frac{1}{4}ffrr = 0$, an equation of the fourth degree, which may be conſtructed after the manner already explained, making uſe of ſuch conical *loci* as ſhall be moſt agreeable. But this equation is diviſible by $y - f$, and the quotient is the equation $y^3 - \frac{3}{4}rry + \frac{1}{4}frr = 0$, which I ſhall conſtruct by a parabola, and an hyperbola between the aſymptotes. Make therefore $yy = rz$, and making the ſubſtitutions, it will be $zy - \frac{3}{4}ry + \frac{1}{4}fr = 0$, an equation to the hyperbola.

Fig. 121.



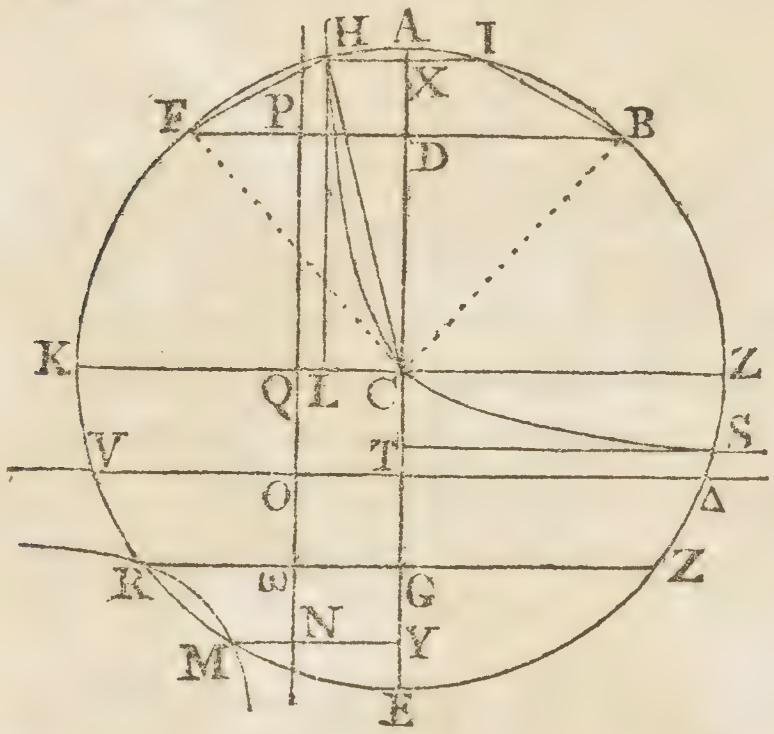
Make AR $= \frac{1}{2}r$, and AB $= \frac{1}{2}f$. Producing AR, AB, each way indefinitely, between them, as aſymptotes, let the hyperbola TPtp be deſcribed, which ſhall paſs through the point O. Then taking RC $= \frac{1}{4}r$, and from the point C drawing the indefinite line CI parallel to AL, take any line whatever, CI $= y$, and it will be IP $= z$, and the hyperbola will be the *locus* of the equation $zy - \frac{3}{4}ry + \frac{1}{4}fr = 0$. With vertex C, diameter CM, and parameter $= r$, let the parabola NCH be deſcribed; it will cut the hyperbola in three points T, P, N, from whence drawing the lines TS, PQ, NM, parallel to AL, theſe ſhall be the three roots of the equation.

It is plain that the parabola will cut the hyperbola TP in the points T, P, becauſe, it being CR $= \frac{1}{4}r$, putting this value inſtead of z in the equation to the parabola, $yy = rz$, it will give us $y = \frac{1}{2}r$. But $\frac{1}{2}r$ is always greater than $\frac{1}{2}f$, and therefore the ordinate in the parabola, which correſponds to the point R, will always be greater than RO; and therefore the parabola will paſs within the hyperbola.

Now, becauſe the circle is given in the Problem, it will be much more convenient to make uſe of this for the conſtruction, by introducing it, firſt, to be added to the final equation, and that by putting the line HL (Fig. 120.) or $\sqrt{rr - yy} = z$. Then it will be DE $= \frac{ay}{z}$, and DF $= 2y + \frac{ay}{z}$, and

therefore the equation is $2y + \frac{ay}{z} = f$, that is, $2yz + ay = fz$, a *locus* to the hyperbola between the asymptotes.

Fig. 122.



Bisecting DF in P, through the point P draw the indefinite line PN parallel to AC, and taking $QO = \frac{1}{2}a$, through the point O draw the indefinite line VΔ parallel to KC. Between the asymptotes PN, VΔ, describe the hyperbola whose rectangle is $\frac{1}{4}af$, which shall pass through the point C; and taking the y's on the line CQ, positive towards the point K, the corresponding ordinates shall be z, and the hyperbola be the *locus* of the equation $2zy + ay - fz = 0$.

This will cut the circle in four points H, R, M, S, from which drawing perpendiculars HX, RG, MY, ST, to AC, these shall be the roots of the equation, three, HX, RG, MY, positive, and one, ST, negative.

It is plain that the root HX, or CL, serves for the division of the given arch FAB; and the root YM serves for the division of FMB, the remainder to the whole circle. For, if I had proposed to divide the arch FMB, I should have had the same equation, and therefore the same *locus*.

The root RG serves to no purpose, but, however, it informs us, that it is equal to f, or that by which the equation is divisible, which results from the two *loci* $rr - yy = zz$, and $2zy + ay - fz = 0$; that is, the solid equation found before.

Now, to demonstrate it, taking $O\omega = \frac{1}{2}a = OQ$, the corresponding ordinate of the circle will be $GR = f$. But $\omega G = PD = \frac{1}{2}f$; therefore $\omega R = \frac{1}{2}f$. But the constant rectangle of the hyperbola is $\frac{1}{4}af$; therefore the hyperbola will cut the circle in the point R, and therefore it will be the root which corresponds to this point.

The other root TS serves for the division of the whole circle into three equal parts, which may be demonstrated in this manner.

Because FD = RG, the arches FK, KR, will be equal; and therefore, RG being produced to Z, the arches FAB, RMZ, will be equal. Therefore FR, or BZ, will be half the difference of the two arches FAB, FMB. But if we should solve the Problem relatively to the arch BZ, we should find the same hyperbola HCS, and ZS would be a third part of the arch BZ, that is, a third part of half the difference of the arches FAB, FMB; and therefore BS is a third part of the said difference. But HB is two-thirds of FAB, and therefore one-third

one-third of the sum of the two arches FAB, RMZ. Therefore the sum of HB and BS, that is, the arch HS, will be a third part of the whole circle.

Q. E. D.

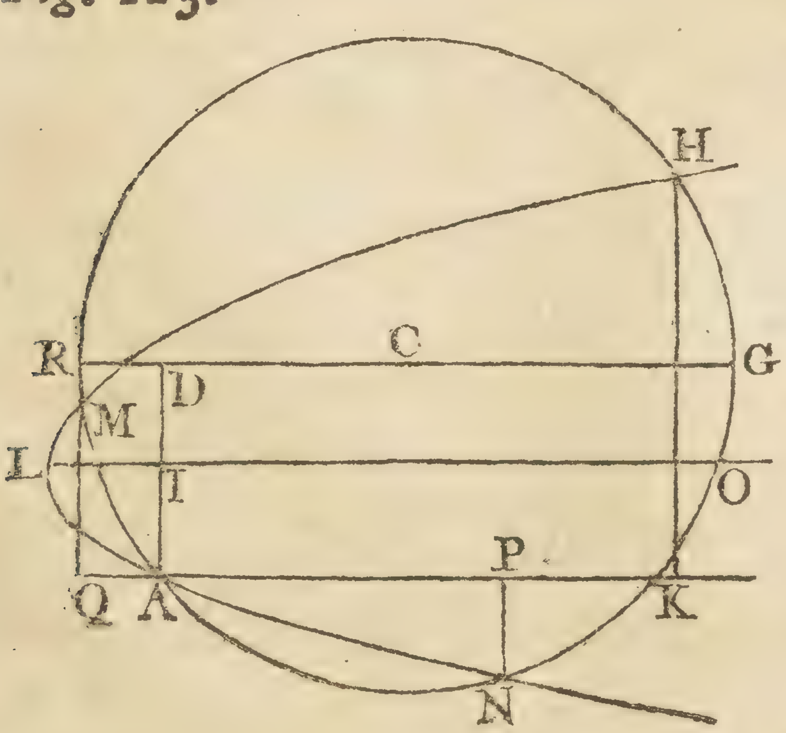
216. This Problem has been resolved before, at § 110, but after another manner. There it is seen, that, in the case wherein the given angle is a right angle, the Problem will be plane. In the other two cases, of an obtuse or acute angle, we arrived at these two cubic equations, $2bx^3 - 3aax^2 + a^4 = 0$, and $2bx^3 + 3aax^2 - a^4 = 0$. Other cases of this Problem constructed.

But if it be considered, that in the first equation, which serves for the obtuse angle, taking x negative, it will be changed into the second, which serves for the acute angle; it will be sufficient to construct the equation for the first case, because the negative root of this will give the solution for the other case.

Therefore I multiply the first equation by $x = 0$, in order to reduce it to the fourth degree, and I divide it by $2b$; then it will become $x^4 - \frac{3aax^3}{2b} + \frac{a^4x}{2b} = 0$.

I take the equation to the parabola $xx - \frac{3aax}{4b} = ay$, and squaring it, it will be $x^4 - \frac{3aax^3}{2b} + \frac{9a^4x^2}{16bb} = aayy$. Then, instead of the two first terms, substituting their value, it will be $yy - \frac{9aaxx}{16bb} + \frac{aax}{2b} = 0$. Here, instead of xx, I substitute it's value $ay + \frac{3aax}{4b}$, and I shall have the equation $yy - \frac{9a^3y}{16bb} - \frac{27a^4x}{64b^3} + \frac{aax}{2b} = 0$; to which adding the first, $xx - \frac{3aax}{4b} - ay = 0$, it will be finally $yy - \frac{9a^3y}{16bb} - \frac{27a^4x}{64b^3} + \frac{a^2x}{2b} + x^2 - \frac{3a^2x}{4b} - ay = 0$, an equation to the circle, taking the co-ordinates at right angles.

Fig. 125.

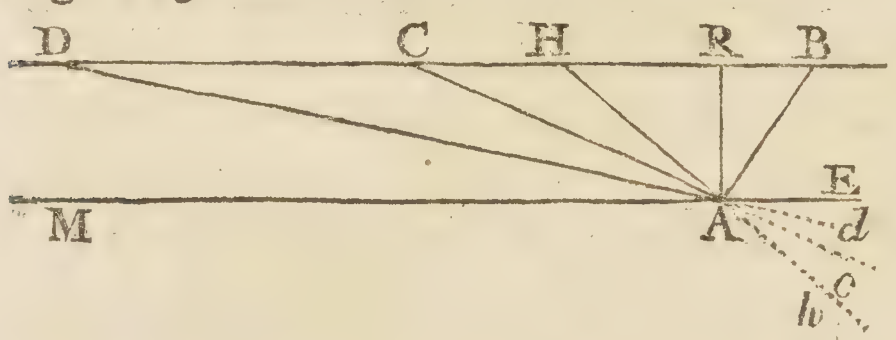


With radius CG $= \sqrt{mm + nn}$, (making, for brevity, $\frac{9a^3 + 16abb}{16bb} = 2m$, and $\frac{27a^4 + 16a^2b^2}{64b^3} = 2n$,) let the circle MNH be described, and taking CD $= m$, from the point D draw DA perpendicular to CD, and equal to n. This will meet the periphery of the circle in the point A. Through this point A draw AK parallel to RG; and, taking any line at pleasure, AK $= y$, the corresponding ordinate will be KH $= x$, and the circle will be the *locus* of the equation.

 On

On the right line AD take $AI = \frac{3aa}{8b}$, and through the point I drawing LO parallel to AK, let there be taken a portion of it, $IL = \frac{9a^3}{64bb}$, and with vertex L, axis LO, and parameter $= a$, let there be deſcribed the *Apollonian* parabola ALH. From the point A taking the abſciſs y on the axis AK, the correſponding ordinates will be $KH = x$, and the parabola will be the *locus* of the equation $xx - \frac{3aax}{4b} = ay$; this will meet the circle in four points, A, M, H, N. The point A will give the introduced root $= 0$. The three perpendiculars, QM, PN, KH, to AK, will give the three roots of the equation. The firſt poſitive root, QM, will ſerve for the obtuſe angle. The ſecond, PN, which is negative, will ſerve for the acute angle. The third, KH, will ſerve for the diviſion, into three equal parts, of that angle which is the difference between the given angle and a right angle.

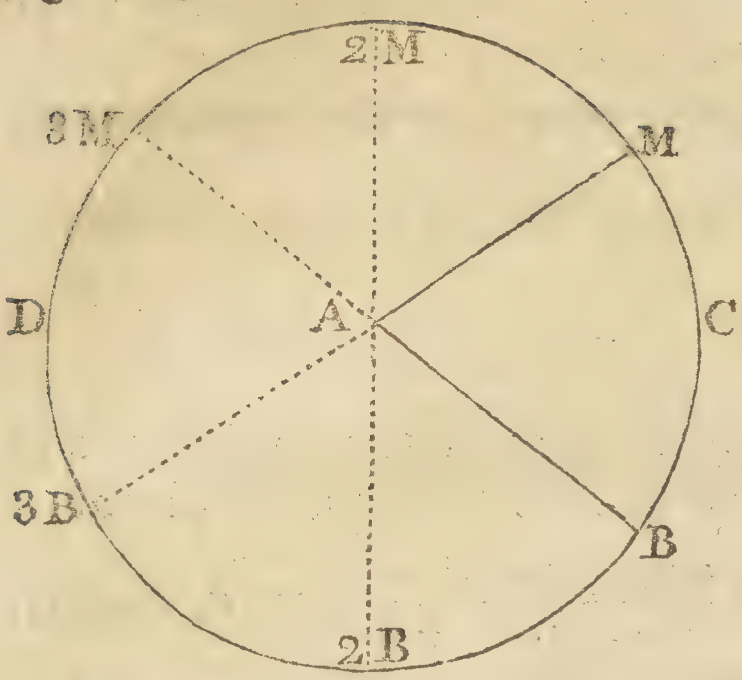
Fig. 123.



Now, to ſhow that this is true, let the given angle be MAB. Let AH be perpendicular to AB; and let us divide the angle MAH into three equal parts, which is the difference between the given angle MAB, and the right angle HAB. Suppoſe it ſo divided by the right lines AC, AD, and repeating the reaſoning of § 110, it will be AC = CD, and the triangle ACH will be ſimilar to the triangle DAH, and therefore we ſhall have the analogy, CH . HA :: HA . DH. Naming the quantities, therefore, as in § 110, $AB = a$, $BR = b$, and $BC = x$, it will be $RC = x - b$, $BH = \frac{aa}{b}$, $CH = x - \frac{aa}{b}$, $AR = \sqrt{aa - bb}$, $HA = \frac{a}{b}\sqrt{aa - bb}$, $AC = \sqrt{aa + xx - 2bx}$, $DH = x - \frac{aa}{b} + \sqrt{aa + xx - 2bx}$. Therefore, ſubſtituting theſe analytical values in the foregoing proportion, it will be $x - \frac{aa}{b} \, . \, \frac{a}{b}\sqrt{aa - bb} :: \frac{a}{b}\sqrt{aa - bb} \, . \, x - \frac{aa}{b} + \sqrt{aa + xx - 2bx}$. Whence the equation $\frac{aa}{bb} \times \overline{aa - bb} = \overline{x - \frac{aa}{b}} \times \overline{x - \frac{aa}{b} + \sqrt{aa + xx - 2bx}}$; which, being reduced, and finally divided by $aa - bb$, will be found to be $2bx^3 - 3aaxx + a^4 = 0$, the very equation which was to be conſtructed.

Beſides the angles leſs than two right ones, which inſiſt on arches leſs than a ſemicircle, and which the architects call *Entrant Angles*, there are alſo angles which are greater than two right ones, and which inſiſt on arches greater than a ſemicircle, and are called *Salient Angles*. The inclination of the two lines AB,

Fig. 124.



AB, AM, which point towards C, may be confidered as pofitive, and that negative which points towards D. As long as the inclination of the two lines AB, AM, fhall be pofitive, and fhall point towards C, fo long the angle MAB fhall be entrant, or lefs than two right angles, and fhall infift upon an arch, BCM, lefs than a femicircle. If the two lines A2B, A2M, fhall make a right line 2B2M, the inclination will be none at all. But if the inclination fhall become negative, the two lines A3B, A3M, winding towards D, then the angle 3MA3B will be changed into a falient angle, greater than two right ones, and will infift upon an arch, 3MC3B, greater than a femicircle. Therefore the trifection of any given angle may alfo include that of a falient angle.

Now it is to be confidered, that, as the line AB (Fig. 123.) infifts upon the line MAE, whilft it forms the angle MAB, three other angles will confequently arife, that is, the entrant BAE, which, united to the given and alfo entrant angle MAB, makes up the two right angles; and the falient angles MAB, BAE, which, united to the correfponding entrant angles, complete the four right angles.

Wherefore the three roots of our equation, $2bx^3 - 3aax^2 + a^4 = 0$, ferve for the trifection of all the fore-mentioned angles. By means of the leaft pofitive root, the obtufe angle MAB is divided into three equal parts; and, by means of the negative, the acute angle BAE, as has been feen. Befides, it has been fhown, that the greater pofitive root ferves for the angle MAH; and this ferves alfo to trifect both the falient angles MAB, BAE. For, indeed, the falient angle BAE is equal to three right angles, together with the angle MAH. The third part, therefore, of the falient angle BAE muft be equal to one right angle, together with the third part of the angle MAH; and fuch is the angle CAB. In like manner, the falient angle MAB is equivalent to three right angles, taking away the angle MAH, or *h*AE; and confequently *c*AB will be it's third part, as being equal to the right angle *h*AB, taking away the angle *h*A*c*, a third part of the angle *h*AE.

217. Now, to divide the given angle into three equal parts, if I had made ufe of Prob. XIII. § 108, I fhould have come to the equation $x^3 - 3bx^2 - 3rrx + brr = 0$; and, multiplying by $x = 0$, it is $x^4 - 3bx^3 - 3rrx^2 + brrx = 0$. Wherefore, affuming the *locus* to the parabola $xx - \frac{1}{2}bx = by$, and doing the reft as ufual, we fhall have another *locus* to the circle, taking the co-ordinates at right angles. This will be $yy - \frac{26b^3y + 24brry}{8bb} - \frac{39b^3x + 28brrx}{8bb} + xx = 0$. The fame conftructed another way.

These

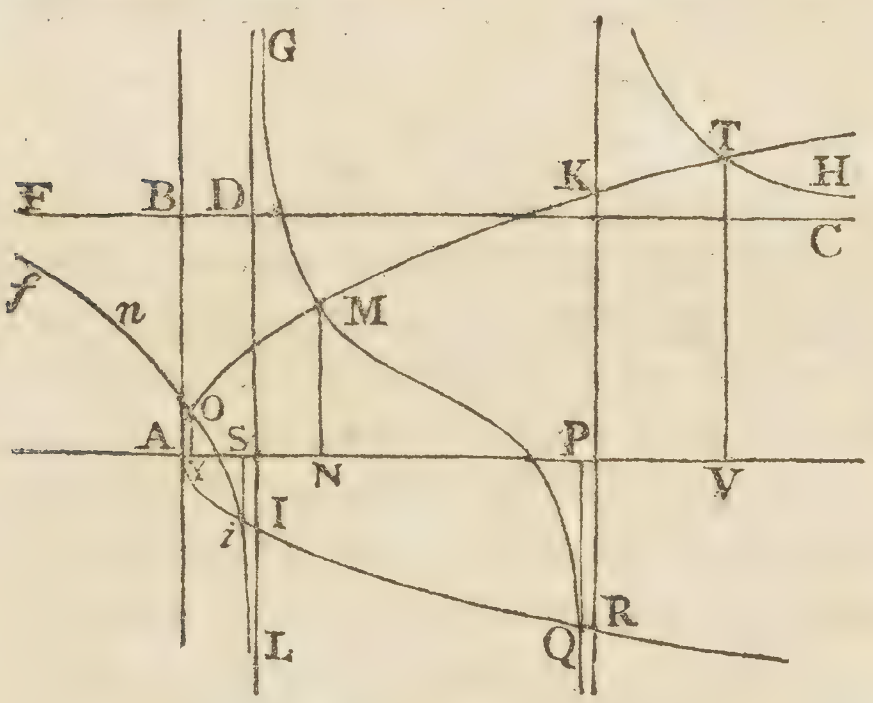
Theſe two *loci* being deſcribed and combined, will give the ſame conſtruction as in Fig. 125, differing only in the known quantities. For, in this caſe, the radius of the circle will be CG $= \sqrt{mm + nn}$; (making, for brevity-ſake, $\frac{26b^3 + 24brr}{8bb} = 2m$, and $\frac{39b^3 + 28brr}{8bb} = 2n$,) and it will be CD $= m$, DA $= n$, AI $= \frac{3}{4}b$, and IL $= \frac{9}{16}b$.

This Problem raiſed higher.

218. From the ſame Problem we have a general method for dividing any given arch or angle into as many equal parts as we pleaſe. Thus, to divide it into five equal parts, we ſhall have this equation, $\frac{5r^4x - 10rrx^3 + x^5}{r^4 - 10rrxx + 5x^4} = b$, that is, $x^5 - 5bx^4 - 10rrx^3 + 10br^2x^2 + 5r^4x - br^4 = 0$.

To conſtruct this, I take a *locus* to the *Apollonian* parabola $xx = ry$, and, making the ſubſtitutions, there ariſes a ſecond of the third degree, $xyy - 5byy - 10rxy + 10bry + 5rrx - brr = 0$, that is, $x = \frac{5byy - 10bry + brr}{yy - 10ry + 5rr}$.

Fig. 126.



Therefore, having deſcribed the *locus* of the equation, which ſhall be the curve with three branches, Fig. 126, that is, HT between the aſymptotes RK, BC; GMQ between the aſymptotes DI, KR; and *fni*L between the aſymptotes DF, DI; in which, on the axis AV, are the y's, and the correſponding ordinates are the x's. With vertex A, parameter $= r$, and axis AV, if the parabola of the equation $xx = ry$ be deſcribed, it will meet the curve in five points, O, M, T, i, Q; which will determine the five roots, or, mn, TV, Si, and PQ; three poſitive, and two negative, of the equation propoſed.

—raiſed ſtill higher.

219. So, to divide an arch or angle given into any greater odd number of equal parts, other curves may be found, relative to the degree of the equation.

SECT.

SECT. V.

Of the Construction of Loci *which exceed the Second Degree.*

220. The Geometrical *Loci* may be constructed after two different manners; that is to say, by describing curves expressing equations which exceed the second degree; if we may call that describing, in each manner, which is rather tracing them out, so as to give some notion of such curves. Higher *loci* constructed two ways.

The first manner of tracing them is, by finding an infinite number of points. The second is, by means of other curves of an inferior degree, which are already described. Thus, a *locus* or equation of the third degree may be constructed by means of a right line and a conic section; a *locus* or equation of the fourth, by means of two conic sections; a *locus* or equation of the fifth, by means of a conic section and a *locus* of the third degree. And so on, as far as you please.

221. Now, as to the first manner, by an infinite number of points; first, the equation must be reduced in such manner, that one of the two unknown quantities, which shall seem fittest for the purpose, must be freed from fractions or co-efficients, must be of one dimension only, and placed alone on one side of the sign of equality; which may always be done by the methods explained in Sect. II. Then, in respect of such unknown quantity, (the other being considered as constant,) the equation must be of it's own nature plane, that is, must not exceed the second degree. As, for example, the equation $xyy + 2aay = x^3$, that is, $yy + \frac{2aay}{x} = xx$, which, managed by the rules for affected quadraticks, will give $y = \frac{-aa \pm \sqrt{x^4 + a^4}}{x}$. —first, by finding an indefinite number of points.

Equations being given or reduced in this manner, the way of constructing the *locus*, or curve expressed by it, consists in giving an arbitrary value to that unknown quantity which is included in the *homogeneum comparationis*; taking it from a fixed point on a right line, which serves as an axis or diameter, according as the angle of the co-ordinates is to be a right or an oblique angle. As in the equation $y = \frac{-aa \pm \sqrt{x^4 + a^4}}{x}$, if we should give to x a value at pleasure, by

that means we ſhould have a congruous value of y alſo. Then, from the extremity of the aſſumed value of x having drawn the value of y, in the given angle of the co-ordinates, this would ſupply us with a point in the curve to be deſcribed. Another value that we may give to the ſame unknown quantity x will ſupply us with another y, and that with another point in the curve; and thus, one after another, by aſſigning different values to x, we ſhall have ſo many y's, or ſo many points of the curve. Now, the greater the number be of theſe points, ſo much the more exact will be the deſcription of the curve, and then only we can have it perfectly exact, when we take an infinite number of ſuch points, at due diſtances.

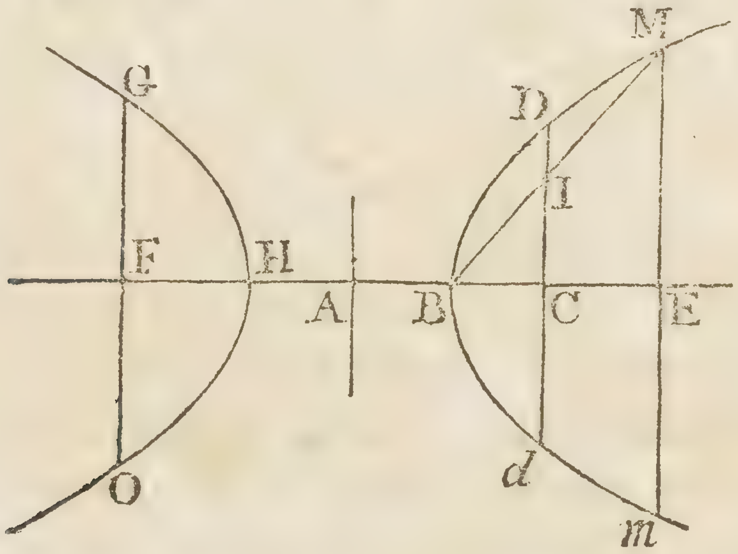
The ordinates to be at right angles to the abſciſs.

222. For the ſake of greater ſimplicity, I ſhall at preſent ſuppoſe, that theſe curves are referred to their axis, or that the angle of the co-ordinates is a right angle; for, in caſe the angle be oblique, no alteration will thence follow.

An Example of deſcribing the curve by points.

223. For the more eaſy underſtanding the application of this method, I ſhall take a ſimple example of a curve already known, that is, of the equilateral hyperbola $yy = xx - aa$, or $y = \pm \sqrt{xx - aa}$.

Fig. 127.



Let A be a fixed point, or the beginning of the x's, to be taken on the indefinite line AE. Firſt, then, I examine what ordinate correſponds to the point A, that is, what will y be when $x = 0$. Therefore, ſubſtituting 0 inſtead of x in the given equation, it will be found $y = \pm \sqrt{0 - aa}$, or y is imaginary and impoſſible. Therefore, to the point A there belongs no point of the curve. By making $x = 0$, if y had not come out imaginary, but only 0, the curve would have begun at the point A. It may be obſerved, that as often as x is leſs than a, the radical $\sqrt{xx - aa}$ will always be negative, and therefore y an imaginary quantity. Therefore, making $AB = a$, to every x leſs than AB an imaginary y will always correſpond, ſo that there will be no point in the curve. I take $x = a = AB$, then $y = \pm \sqrt{aa - aa} = 0$; and therefore B will be a point in the curve, or rather, the curve will have it's origin in the point B. I take $x = 2a = AC$, and it will be $y = \pm \sqrt{4aa - aa} = \pm \sqrt{3a^2}$, poſitive and negative. Therefore make CD poſitive, and Cd negative, each equal to $\sqrt{3aa}$, and D and d will be two points in the curve. I take $x = 3a = AE$, and it will be $y = \pm \sqrt{8aa}$. Making therefore EM poſitive, and Em negative, $= \sqrt{8aa}$, and M, m, will be two points in the curve. And thus going on continually, by giving other values to x, we ſhall have the congruous values of y. And it is eaſy to perceive, that, as the x's increaſe, ſo the

 quantities

quantities $\sqrt{xx - aa}$ will perpetually increaſe, that is, the values of y, both affirmative and negative. Thus, the curve will always proceed, enlarging and lengthening itſelf both above and below the axis; and, laſtly, taking x infinite, becauſe, to ſubtract a finite quantity from one that is infinite, is the ſame thing as to ſubtract nothing; therefore $\sqrt{xx - aa}$ will become $\sqrt{xx}$, or x, and we ſhall have $y = \pm x$, and y poſitive and negative will be infinite, and therefore the curve will go on *ad infinitum*.

224. And becauſe, in the equation $y = \pm \sqrt{xx - aa}$, the unknown quantity x is raiſed to an even power, that is, to the ſquare; if we take x negative, the equation itſelf receives no alteration. Hence it is, that, if we aſſign negative values to x, or if we take it on the ſide of A towards F, the ſame curve would be deſcribed as before, but in a contrary poſition with it's vertex H, it being AH = AB. And to no-abſciſs x, poſitive or negative, taken between B and H, any real ordinate, poſitive or negative, will correſpond; that is, there will be no point of the curve.

In even powers, the ſign of the axis is ambiguous.

225. Now it is plainly ſeen, that the given curve cuts the axis in no point out of the vertices B, H; for, as x increaſes, y always increaſes. Nevertheleſs, it very often happens, that, beſides the vertex, they cut it in other points, in which caſe y muſt neceſſarily become nothing. Therefore, to have theſe points, in the given equation we muſt ſuppoſe $y = 0$, and find the values of x on this ſuppoſition, which will give us the points required. Wherefore, in the equation $yy = xx - aa$, ſuppoſing $y = 0$, it will be $xx = aa$, that is, $x = \pm a$. Therefore, in the points B, H, only, the curve will cut the axis, and not in any other.

To find where the curve cuts the axis.

226. If, between the points B, C, other values of x be taken, we ſhall alſo have the correſponding values of y, that is, other points of the curve between B and D, as alſo, between B and d; ſo that the more points we have, the more exact will be the deſcription of the part BD, or Bd; but we can never have it perfect, unleſs the number of thoſe points were infinite. And the ſame may be ſaid of any other portion.

The more points we take, the better.

227. Now it is plain, that if either of the two indefinite quantities be made infinite, and the other be neither infinite nor imaginary, but be either finite or equal to nothing, the firſt indeterminate will be an aſymptote to the curve, which will correſpond to ſome determinate point of the value of the ſecond. Therefore, to inquire if a curve have aſymptotes, and where they are, it will ſuffice to make y infinite, and to ſee what value for x will then reſult from the equation. Then, to make x infinite, and ſee what value for y will thence reſult. In the equation $y = \pm \sqrt{xx - aa}$, making y infinite, it will be $\sqrt{xx - aa} = \infty$, and therefore $xx - aa = \infty$, or $xx = \infty$, and therefore x is infinite; for the root of an infinite ſquare muſt alſo be infinite. So that y cannot be infinite

To find when a curve can have an aſymptote.

 except

except when x is infinite alſo; and therefore the axis of the y's cannot be an aſymptote. Making x infinite, $\sqrt{xx - aa}$ will be the ſame; for a finite quantity, added to or taken from an infinite quantity, can make no alteration; it will be now $y = \pm x$, or, if x be infinite, y will be ſo alſo, and it's axis cannot be an aſymptote.

—found by changing the equation.

228. But it is not ſo in the equation $ay + xy = bb$, which we otherwiſe know to belong to the hyperbola between it's aſymptotes. For, taking y infinite, the two terms $ay + xy$ will be infinite, and, in reſpect of them, the term bb will be nothing, and the equation will become $ay + xy = 0$, and, dividing by y, it is $x = -a$; ſo that, taking $x = -a$, the ordinate, which in this point is infinite, will be an aſymptote to the curve. Then, taking x infinite, becauſe the two rectangles ay, xy, having the ſame altitude y, are to each other as their baſes a, x, the ſecond muſt be infinitely greater than the firſt, or ay will be nothing in reſpect of xy. Therefore, expunging ay out of the equation, there will remain $xy = bb$, or $y = \frac{bb}{x}$. But x is infinite by ſuppoſition, therefore $y = \frac{bb}{\infty} = 0$. So that when $y = 0$, then x will be infinite, and therefore is an aſymptote to the curve.

Cautions to be obſerved in finding aſymptotes.

229. But here it muſt be obſerved, that this way of arguing takes place only in the caſe of aſymptotes parallel to the co-ordinates, and not otherwiſe. For the truth is, the hyperbola $yy = xx - aa$ has indeed it's aſymptotes, but which are not parallel to either of the co-ordinates; therefore, in this caſe, the preſent way of arguing cannot be applied, but there is need of a further artifice; which, as it depends on the Method of Infiniteſimals, I ſhall reſerve for another place.

To find if the curve be concave or convex towards it's axis.

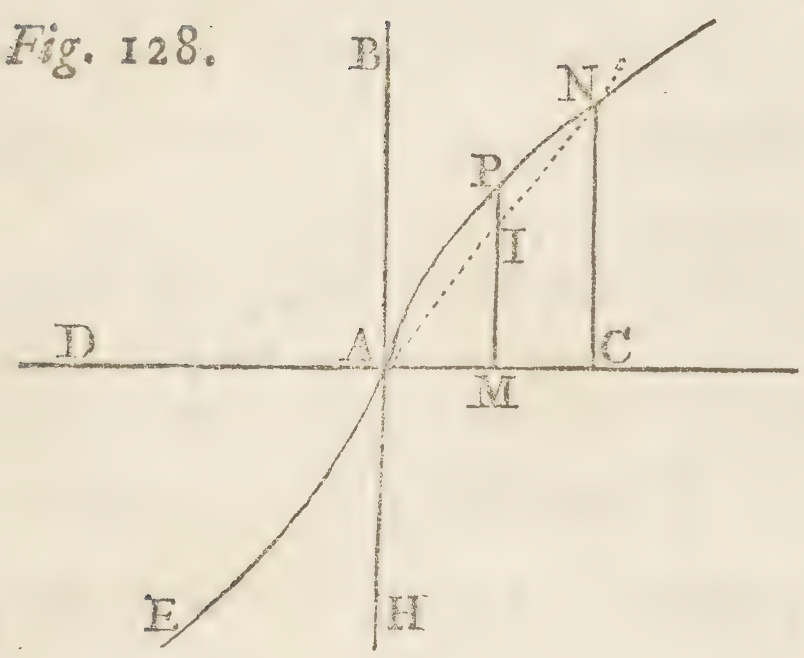
230. It remains to inquire, whether the ſaid curve $y = \pm\sqrt{xx - aa}$ be concave or convex towards it's axis; for which purpoſe, we muſt take from it's origin any abſciſs AE of a determinate value, and, by means of the given equation, we muſt find the value of the correſponding ordinate EM. Then, taking another abſciſs AC of a determinate value leſs than the former, we muſt find the value of the correſponding ordinate CD; and drawing the right line BM, which ſhall cut CD (produced, if occaſion) in I; and the lines AE, AC, being known, or BE, BC, and the ordinate EM, by the ſimilar triangles BEM, BCI, we ſhall find the value of CI; and if this be leſs than CD, the curve will be concave towards the axis AE, as is plain; but if it be greater, the curve will be convex. In the given Example, I take $x = \text{AE} = 3a$; then $y = \sqrt{8a^2}$. Again, I take $x = \text{AC} = 2a$; then $y = \text{CD} = \sqrt{3aa}$. Now, becauſe $\text{BE} = 2a$, $\text{BC} = a$, it will be $\text{CI} = \frac{\sqrt{8aa}}{2} = \sqrt{2aa}$, that is, CI leſs than CD, and therefore the curve is concave towards the axis AE.

231. But

231. But these conclusions are valid only in such curves, which have no point of contrary flexure, or of regression. But, because these have their particular methods, of which, at present, this is not a place to treat, we cannot as yet form a just and complete idea of such curves. Further to determine the forms of the curves, with examples.

EXAMPLE II.

Fig. 128.



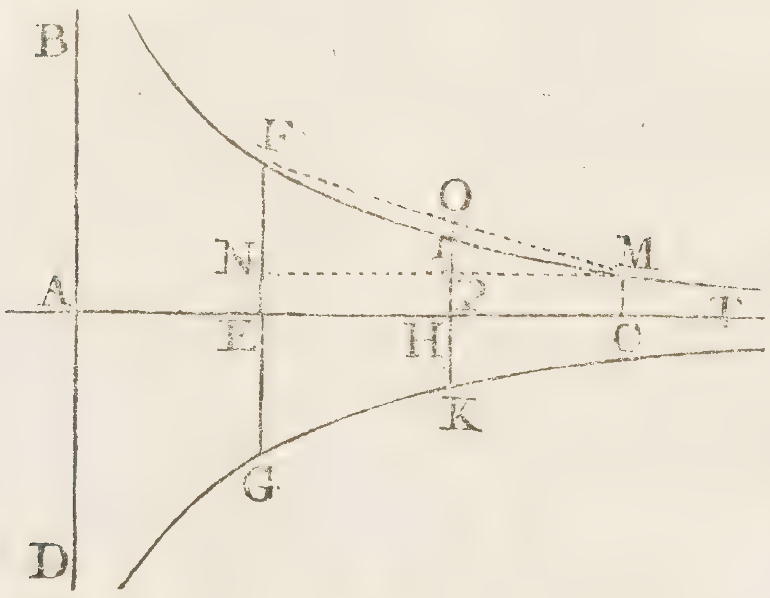
Let the equation be $y^3 = aax$, or $y = \sqrt[3]{aax}$. Drawing the two indefinite lines BH, DC, making a given angle BAC equal to that of the co-ordinates; in AC, from the point A let the x's be taken, and the y's upon AB, or a line parallel to AB. First, I inquire if the curve passes through the point A or not, that is, what will y be when $x = 0$. But, making $x = 0$, we find $y = \sqrt[3]{aa \times 0}$, that is, $y = 0$. Therefore the curve passes through the point A. I inquire further, if the curve cuts the axis AC in another point, that is to say, what is x when $y = 0$, and I find $\sqrt[3]{aax} = 0$, that is, $x = 0$. Therefore the curve cuts the axis in no other point but A. Make $x = AM = \frac{1}{2}a$, and the given equation will be $y = \sqrt[3]{\frac{1}{2}a^3}$. Therefore, drawing $MP = \sqrt[3]{\frac{1}{2}a^3}$, and parallel to AB, then P will be a point in the curve. I make $x = AC = a$, and it will be $y = \sqrt[3]{a^3} = a$; then drawing $CN = a$, and parallel to AB, N will be another point in the curve. And doing this successively, we may find as many points as we please, through which the curve of this equation will pass. Lastly, make x infinite, or $x = \infty$, and it will be $y = \sqrt[3]{aa \times \infty}$, that is, y is infinite, and therefore our curve passes on to infinity. And because, taking $x = 0$, it is also $y = 0$, and taking $x = \infty$, it is also $y = \infty$, the curve will have no asymptotes that are parallel to the co-ordinates.

Let the line AN be drawn beneath, which cuts in I the line MP, produced if necessary. Now, since $AM = \frac{1}{2}a$, $AC = a = CN$, it will be $MI = \frac{1}{2}a$. But $MP = \sqrt[3]{\frac{1}{2}a^3}$, therefore MI will be less than MP, and therefore the curve is concave to the axis AC.

Now, if we take the abscis negative, because in the given equation $y^3 = aax$, the exponent of x is odd, when x is taken negative it's sign should be changed, and the equation will then be $y = \sqrt[3]{-aax}$; here it is evident, that, taking the values of x the negative way, that is, from A towards D, but equal to those already taken the positive way, it will give as many negative values of y, equal to the positive. Whence the branch AE will be just the same as the branch AN, but contrarily posited.

EXAMPLE III.

Fig. 129.



Let the equation be $a^3 - zyy = 0$, that is, $y = \pm\sqrt{\frac{a^3}{z}}$, and let us take the z's from the point A on the axis AC. First, I inquire if the curve passes through the point A; making therefore $z = 0$, the equation will be $y = \pm\sqrt{\frac{a^3}{0}}$, that is, $y = \pm\infty$. Therefore BD, being infinite on both sides of A, will be an asymptote to the curve. Next, I inquire if in no point the curve cuts the axis; and therefore put $y = 0$, and the equation will be $\pm\sqrt{\frac{a^3}{z}} = 0$, or $\frac{a^3}{z} = 0$, or $z = \frac{a^3}{0}$, that is, $z = \infty$. Therefore AC will be another asymptote. Taking $z = a = $ AE, it will be $y = \pm\sqrt{\frac{a^3}{a}} = \pm a$. Making therefore EF positive and EG negative, and each $= a$, the points F, G, will be in the curve. Taking $z = 2a$ $=$ AH, it will be $y = \pm\sqrt{\frac{a^3}{2a}} = \pm\sqrt{\frac{1}{2}aa}$. Therefore, making HI positive, and HK negative, each equal to $\sqrt{\frac{1}{2}aa}$, the points I, K, will be in the curve. Taking new values of z always greater and greater continually, there will result new values of y always less and less, so that the two branches, FI, GK, of the curve being in every thing equal and similar, will stretch out on each side, approaching to the asymptotes BD, AC, yet without ever touching them, but at an infinite distance from the point A.

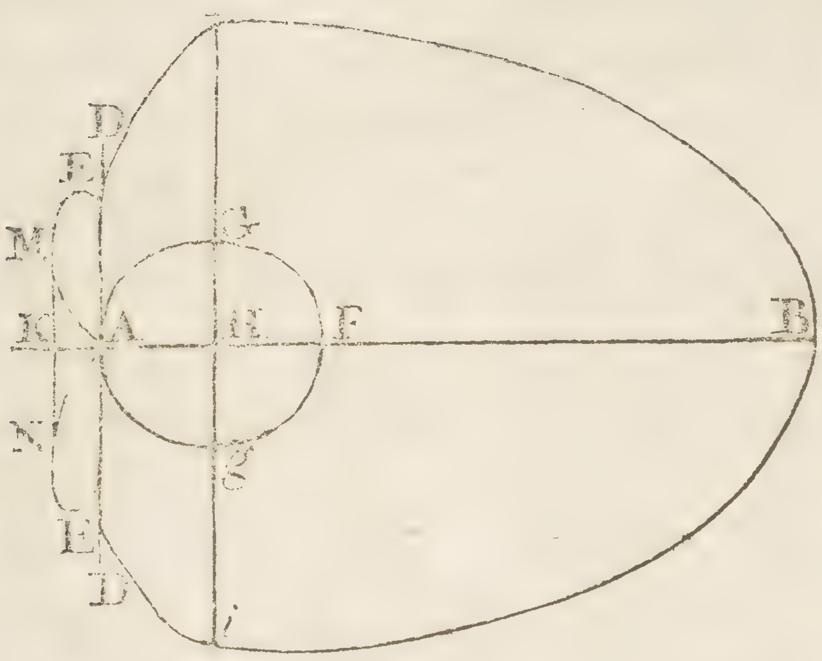
As to the negative abscifs z; because the exponent of z is an odd number, if it be taken negative it will be convenient to change the sign of the term $-zyy$, and then the equation will be $a^3 + zyy = 0$; that is, $y = \pm\sqrt{-\frac{a^3}{z}}$. That is to say, the ordinate y is imaginary, and therefore on the negative part of the abscifs there will be no curve.

To examine whether the curve be concave or convex towards it's axis AC, I take AC $= 3a$; then it will be CM $= \sqrt{\frac{1}{3}aa}$; and drawing FM, which cuts HI (produced, if occasion) in O, and MN parallel to AC, it will be NF $=$ $a - \sqrt{\frac{1}{3}aa}$, PI $= \sqrt{\frac{1}{2}aa} - \sqrt{\frac{1}{3}aa}$. Then making the analogy, MN . NF :: MP . PO, that is, $2a \,.\, a - \sqrt{\frac{1}{3}aa} :: a \,.\,$ PO; it will be PO $= \frac{a - \sqrt{\frac{1}{3}aa}}{2}$; and

and therefore, if PO be greater than PI, the curve will be convex towards the axis AC. This is to be examined thus. If it be $\frac{a - \sqrt{\frac{1}{3}aa}}{2} > \sqrt{\frac{1}{2}aa} - \sqrt{\frac{1}{3}aa}$, then multiplying by 2, it will be $a - \sqrt{\frac{1}{3}aa} > 2\sqrt{\frac{1}{2}aa} - 2\sqrt{\frac{1}{3}aa}$, and $a + \sqrt{\frac{1}{3}aa} > 2\sqrt{\frac{1}{2}aa}$, and squaring, $aa + 2a\sqrt{\frac{1}{3}aa} + \frac{1}{3}aa > 2aa$, and multiplying by 3, $3aa + 6a\sqrt{\frac{1}{3}aa} + aa > 6aa$, and reducing the terms, $6a\sqrt{\frac{1}{3}aa} > 2aa$, and dividing by $2a$, $3\sqrt{\frac{1}{3}aa} > a$, and, lastly, squaring, $\frac{9}{3}aa > aa$, or $3 > 1$. Now, as this is a true result, so it is also true PO is greater than PI, and consequently the curve is convex towards the axis AT.

EXAMPLE IV.

Fig. 130.



Let the equation of the curve be $y = \pm\sqrt{\frac{4ax + a^2 - 2x^2 \pm a\sqrt{a^2 + 8ax}}{2}}$. On the indefinite right line AB, taking the x's from the fixed point A, and the y's on AD, which makes the angle DAB of the co-ordinates; if it be put $x = 0$, it will be $y = \pm\sqrt{\frac{aa \pm a\sqrt{aa}}{2}}$, that is, $y = \pm\sqrt{\frac{2aa}{2}}$, and $y = \pm\sqrt{\frac{0}{2}}$; or $y = \pm a$, and $y = 0$. Therefore, making AE positive and negative $= a$, the points E, A, E, will be in the curve. To find where the curve cuts the axis AB, I put $y = 0$, and therefore $\pm\sqrt{\frac{4ax + a^2 - 2x^2 \pm a\sqrt{a^2 + 8ax}}{2}} = 0$. Then, squaring and transposing, $4ax + aa - 2xx = \pm a\sqrt{aa + 8ax}$, and squaring again, $16aaxx + 8a^3x + a^4 + 4x^4 - 16ax^3 - 4aaxx = a^4 + 8a^3x$; then, reducing and dividing by $4xx$, it is $3aa - 4ax + xx = 0$, and resolving the equation, $x = \pm a + 2a$, that is, $x = a$, and $x = 3a$. Therefore, taking $x = AF = a$, and $x = AB = 3a$, the curve will cut the axis in the points F, B. Make $x = \frac{1}{2}a = AH$, it will be $y = \pm\sqrt{\frac{5aa \pm 2a\sqrt{5aa}}{4}}$; therefore the four values of y are real, because $2a\sqrt{5aa}$ is less than $5aa$; which roots are, $\sqrt{\frac{5aa+2a\sqrt{5aa}}{4}}$, $\sqrt{\frac{5aa-2a\sqrt{5aa}}{4}}$, $-\sqrt{\frac{5aa - 2a\sqrt{5aa}}{4}}$, and $-\sqrt{\frac{5aa + 2a\sqrt{5aa}}{4}}$. The two positive roots are relatively

relatively equal to the two negative; therefore, taking HI $=$ Hi $= \sqrt{\frac{5aa + 2a\sqrt{5aa}}{4}}$, and HG $=$ Hg $= \sqrt{\frac{5aa - 2a\sqrt{5aa}}{4}}$, the four points, I, G, g, i, will be in the curve.

Examples to determine when the ordinates are real.

232. As often as the quantity under the common radical vinculum is a negative quantity, (for that under the fecond vinculum, or $\sqrt{aa + 8ax}$, cannot be negative, the abfcifs being pofitive, as I now fuppofe it,) the ordinate y will be imaginary. Now, therefore, that there may be an ordinate, it will be neceffary that it be $\sqrt{\frac{4ax + aa - 2xx \pm a\sqrt{aa + 8ax}}{2}} > 0$.

In the firft place, I take the fign pofitive of the fecond radical, in which cafe the whole quantity will be certainly pofitive, if it be $4ax + aa - 2xx > 0$, that is, $2xx - 4ax < aa$, and therefore $xx - 2ax < \frac{1}{2}aa$, and $xx - 2ax + aa < \frac{3}{2}aa$, and extracting the root, $x - a < \sqrt{\frac{3}{2}aa}$, or $a - x < \sqrt{\frac{3}{2}aa}$. From the firft root, in which x is fuppofed to be greater than a, I infer that it muft be $x < a + \sqrt{\frac{3}{2}aa}$. From the fecond, in which it is fuppofed that $x < a$, I conclude that it muft be $x > a - \sqrt{\frac{3}{2}aa}$. But, as $a - \sqrt{\frac{3}{2}aa}$ is always a negative quantity, it will be always $x > a - \sqrt{\frac{3}{2}aa}$, when x is taken lefs than a. Therefore, taking x lefs than a, the quantity $4ax + aa - 2xx$ will be pofitive, fo that much more the quantity $4ax + a^2 - 2x^2 + a\sqrt{a^2 + 8ax}$ will be pofitive. And therefore, in general, taking x lefs than AF, or a, it will be $y = \pm\sqrt{\frac{4ax + a^2 - 2x^2 + a\sqrt{a^2 + 8ax}}{2}}$, a real ordinate. But, even though $4ax + aa - 2xx$ were a negative quantity, yet $\sqrt{\frac{4ax + aa - 2xx + a\sqrt{aa + 8ax}}{2}}$ may be a pofitive quantity; that is, whenever it is $\sqrt{\frac{4ax + aa - 2xx + a\sqrt{aa + 8ax}}{2}} > 0$, it will be, by fquaring and tranfpofing, $a\sqrt{aa + 8ax} > 2xx - aa - 4ax$, and by fquaring again, $a^4 + 8a^3x > 4x^4 - 16ax^3 + 16a^2x^2 - 4a^2x^2 + 8a^3x + a^4$, that is, $4x^4 - 16ax^3 + 12aaxx < 0$, and dividing by $4xx$, it is $xx - 4ax + 3aa < 0$, and therefore $xx - 4ax + 4aa < aa$, and extracting the root, $x - 2a < a$, as alfo $2a - x < a$. From the firft root, which fuppofes x to be greater than $2a$, arifes $x < 3a$. Therefore, taking x greater than $2a$, but lefs than AB, or $3a$, the radical will be pofitive, and therefore the ordinate y will be real. From the fecond root, which fuppofes x lefs than $2a$, I obtain $x > a$; and therefore, whenever x is greater than a, and lefs than $2a$, the radical will be pofitive, and therefore y real. But we have feen by the firft, that, taking x lefs than a, the ordinate y is real; therefore, in general, the ordinate y will be real, if we take x lefs than AB, or $3a$.

Taking

Taking the sign negative of the second radical, it would be $\sqrt{\frac{4ax + aa - 2xx - a\sqrt{aa + 8ax}}{2}} > 0$, and squaring, $4ax + aa - 2xx > a\sqrt{aa + 8ax}$, and squaring again and reducing, and dividing by $4xx$, it will be $xx - 4ax > -3aa$, and thence also $xx - 4ax + 4aa > aa$, and extracting the root, $x - 2a > a$, as also, $2a - x > a$. From the first root we obtain $x > 3a$. But we have seen, that $x > 3a$ gives the value of y imaginary, when the second radical has a positive sign, and therefore much more when it has a negative sign. Wherefore, omitting this root, I make use of the other, $2a - x > a$, which gives me $x < a$. Therefore, taking x less than AF, or a, the quantity under the common radical vinculum will be positive, as well if we take the sign of the second radical positive as negative, and therefore between A and F there will correspond four real ordinates, that is, two positive and two negative, which are relatively equal to the positive. But when x is greater than AF, or a, the negative sign of the second radical gives an imaginary ordinate, and the positive sign gives it real; because it is x less than AB, or $3a$, and therefore between F and B will correspond to the same abscis only two real ordinates, one positive, the other negative and equal to the positive; and beyond the point B they will be only imaginary.

Now let the abscisses be taken negative, that is, from A towards K. In this case, changing in the equation the signs of all the terms in which the exponent of x is odd, then $y = \pm\sqrt{\frac{aa - 2xx - 4ax \pm a\sqrt{aa - 8ax}}{2}}$. I put $x = 0$, and it will be $y = \pm\sqrt{\frac{aa \pm a\sqrt{aa}}{2}}$, that is, $y = \pm a$, and $y = 0$. Therefore the points E, A, E, will be in the curve, as in the first case. To see if the curve cuts the axis, put $y = 0$; then $\sqrt{\frac{aa - 2xx - 4ax \pm a\sqrt{aa - 8ax}}{2}} = 0$, and squaring, and transposing, $aa - 2xx - 4ax = a\sqrt{aa - 8ax}$, and squaring again, and reducing, and dividing by $4xx$, it will be $xx + 4ax + 3aa = 0$; and resolving, $x = -2a \pm a$.

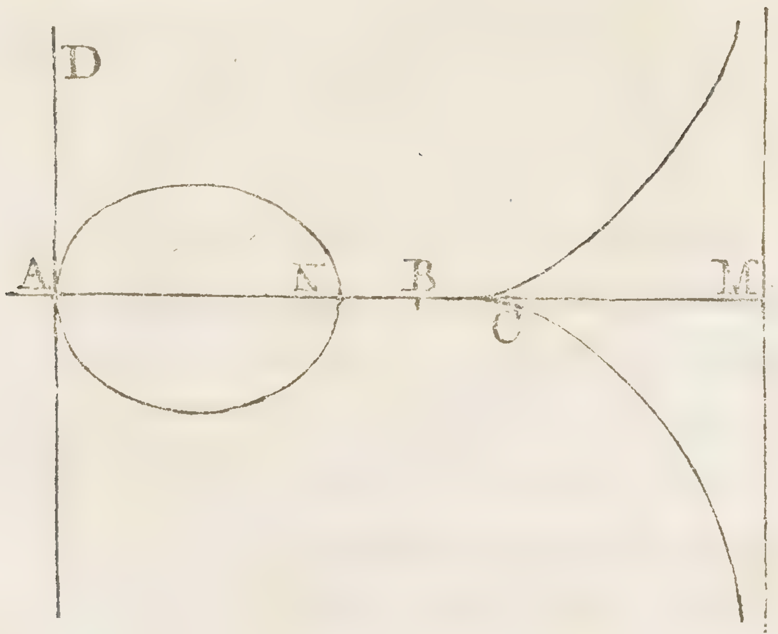
Therefore the curve will cut the axis when it is $x = 0$, a division being just now made by $4xx$; when it is $x = -3a$, and when it is $x = -a$; that is, by being a negative quantity, on the side opposite to this, towards which we now take x; and therefore only in A, F, B, as has been already seen. Now I put $x = \infty$, to see if the curve goes on to infinity, or to the asymptote AK, and it is $y = \pm\sqrt{-2 \times \infty^2 \pm \sqrt{-8a \times \infty}}$, that is, y is imaginary. I inquire then what are the limits of the real ordinates. It is certain that then x is greater than $\frac{1}{8}a$; the second radical will be a negative quantity, and therefore the ordinate y imaginary; so that x must not be taken greater than $\frac{1}{8}a$; but in this hypothesis, because the whole quantity under the common radical is positive, taking the positive sign of the second radical, it will be enough that $aa - 2xx$

7 $- 4ax$

$-4ax$ be positive, that is, $aa - 2xx - 4ax > 0$, and therefore $xx + 2ax < \frac{1}{2}aa$, or $x < \sqrt{\frac{3}{2}aa} - a$. But when x is not greater than $\frac{1}{8}a$, and also $< \sqrt{\frac{3}{2}aa} - a$, making then x not greater than $\frac{1}{8}a$, the ordinate will be real. Taking the negative sign of the second radical, it will be $\sqrt{\frac{aa - 2xx - 4ax - a\sqrt{aa - 8ax}}{2}} > 0$, that is, squaring and transposing, $aa - 2xx - 4ax > a\sqrt{aa - 8ax}$, and squaring again and reducing, $x + 2a > a$. But $x + 2a$ is always greater than a, and therefore, supposing x to be taken not greater than $\frac{1}{8}a$, the ordinates will always be real. I take $x = \frac{1}{8}a$, and it will be $y = \pm\frac{\sqrt{15aa}}{8}$; and therefore, making KM positive, and KN negative and equal to $\frac{\sqrt{15aa}}{8}$, the points M, N, will be in the curve. I take $x = \frac{a}{16}$, it will be $y = \pm\frac{\sqrt{95aa \pm 128a\sqrt{\frac{1}{2}aa}}}{16}$, that is, the four values are real, two positive, which are relatively equal to the two negative. And, because the fourth proportional of $\frac{1}{8}a$, $\frac{\sqrt{15aa}}{8}$, and $\frac{1}{16}a$, or $\frac{\sqrt{15aa}}{16}$, is less than $\frac{\sqrt{95aa + 128a\sqrt{\frac{1}{2}aa}}}{16}$, but greater than $\frac{\sqrt{95aa - 128a\sqrt{\frac{1}{2}aa}}}{16}$, the curve will have two branches above AK, one concave, and the other convex, and also two below, like and equal to those above, as in Fig. 130.

EXAMPLE V.

Fig. 131.



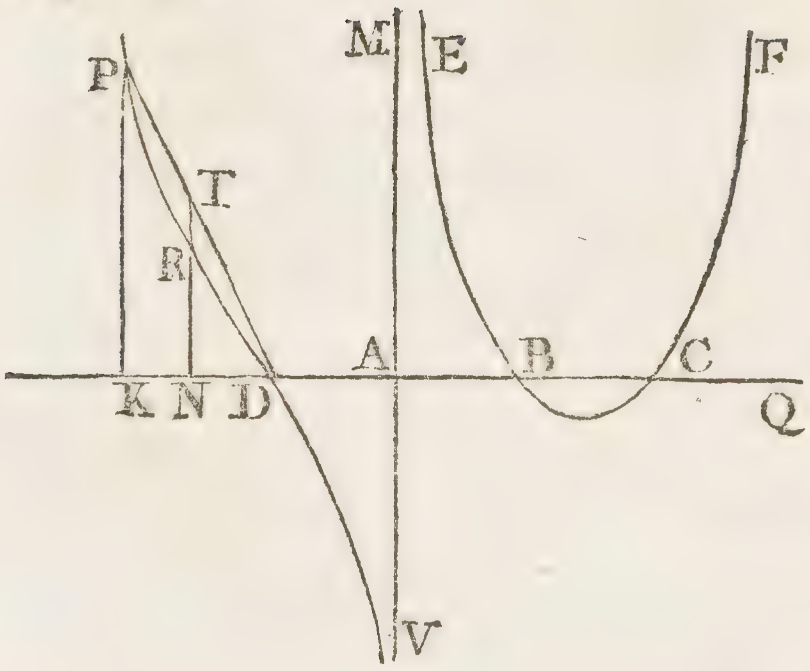
Let it be the curve of this equation $y = \pm\sqrt{\frac{bbxx - x^3 + 2ax^2 - aax}{x - 2a}}$; here, for one case, let a be greater than b, and let the x's be taken from the point A, upon the indefinite line AM, and the y's upon AD in a given angle, or parallel to a given line. Making $x = 0$, it will be $y = 0$, and therefore the point A is in the curve. Making $y = 0$, it will be $\sqrt{\frac{bbx - x^3 + 2axx - aax}{x - 2a}} = 0$, that is, $bbx - x^3 + 2axx - aax = 0$, and dividing by x, it is $bb - xx + 2ax - aa = 0$, and therefore $xx - 2ax + aa = bb$, and extracting the root, $x - a = \pm b$; therefore the values of x will be $x = a + b$, $x = a - b$, and $x = 0$, because the equation was divided by x. Whence, making $AB = BM = a$, BN

BN $= $ BC $= b$, the curve will cut the axis in the point A, as has been already ſeen, and alſo in the points N, C. Making $x = $ AM $= 2a$, y will be poſitive and negative infinite, and therefore there will be an aſymptote at M. Put $x = \infty$, it will be $y = \pm \sqrt{-xx}$, that is, imaginary. Therefore the curve is not continued to infinity. Now, that the ordinate y may be real, it follows that the quantity under the vinculum muſt be poſitive; it is therefore neceſſary that, the numerator of the fraction being poſitive, the denominator muſt be ſo alſo; and the one being negative, the other muſt be the ſame. But, that the numerator may be poſitive, it muſt be $bbx - x^3 + 2axx - aax > 0$, or, dividing by x and tranſpoſing, $xx - 2ax < bb - aa$. Therefore $xx - 2ax + aa < bb$, and extracting the root, $x - a < b$, taking x greater than a; and $a - x < b$, taking x leſs than a. From the firſt root, $x - a < b$, we have $x < a + b$. From the ſecond, $a - x < b$, we have $x > a - b$. Therefore, taking x greater than a, it muſt be $x < a + b$; and taking x leſs than a, it muſt be $x > a - b$, ſo that the numerator may be poſitive. Now, that the denominator may be poſitive, it muſt be $x > 2a$; and, as it cannot be greater than $2a$, and at the ſame time leſs than $a + b$, and than a, the numerator and denominator cannot be both poſitive; and therefore between the points N and C there will be no real ordinates. If we take $x > a + b$, the numerator will be negative; as alſo, if we take $x < a - b$. And if we take $x < 2a$, the denominator will alſo be negative. Therefore, between A and N, and between C and M, there will be real ordinates, and the curve will be nearly as in Fig. 131.

Take x negative; changing therefore the ſigns of thoſe terms, in which the exponent of x is an odd number, the equation will be $y = \pm \sqrt{\frac{x^3 - bbx + 2axx + aax}{-2a - x}}$, that is, $y = \pm \sqrt{\frac{bbx - x^3 - 2axx - aax}{2a + x}}$. The denominator will always be poſitive; but, that the numerator may be poſitive, it will be neceſſary that $b^2x - x^3 - 2ax^2 - a^2x > 0$; and, dividing by x and tranſpoſing, $xx + 2ax + aa < bb$, that is, $x + a < b$, and therefore $x < b - a$. But, if we ſuppoſe $b < a$, then $b - a$ will be a negative quantity, and therefore it can never be $x < b - a$, that is, the numerator can never be poſitive. So that the ordinates y will always be imaginary, and there can be no part of the curve on the ſide of the negative abſciſſes.

EXAMPLE VI.

Fig. 132.



Let the equation be $y^3 - 2ay^2 - aay + 2a^3 = axy$, that is, $x = \frac{y^3 - 2ay^2 - aay + 2a^3}{ay}$. From the fixed point A, upon the indefinite line AQ, I take the y's, and on the indefinite line AM, or it's parallel, in the given angle of the co-ordinates, I take the x's. Putting $y = 0$, it will be $x = \frac{2aa}{0}$, that is, $x = \infty$; ſo that the curve will approach to the aſymptote AM. To ſee if the curve cuts the axis, and where, I put $x = 0$, and therefore $y^3 - 2ay^2 - aay + 2a^3 = 0$; and, reſolving this cubic equation, we ſhall have three values of y, that is, $y = a$, $y = 2a$, and $y = -a$. Therefore, making AB $=$ AD $=$ BC $= a$, the curve will cut the axis in the points B, C, on the ſide of the poſitives, and in the point D on the negative ſide.

To determine the ſame when the equations are irreducible.

233. If the equation $y^3 - 2ay^2 - aay + 2a^3 = 0$ had been irreducible, ſo that we could not have had the analytical values of y, we muſt have conſtructed this equation, and by that means have found the values of y geometrically, that is, expreſſed by lines, which would have given us the points required. And this is to be underſtood of any other ſuch caſe. Thus, I put $y = \frac{3}{2}a$, and it will be $x = -\frac{5}{12}a$, that is, the ordinate is negative, and therefore the curve paſſes below the axis AQ at B, and returns above at C. I put $y = \infty$, it will be $x = \frac{yy}{a} = \infty$, and therefore the curve goes on to infinity. It is plain that the infinite branch BE will be convex towards the axis AM, the branch BC will be concave to the axis AQ, and CF convex, when the curve ſhall have no contrary flexures. Let us now take the abſciſſes y negative from A towards D. Then the equation will be $x = \frac{-y^3 - 2ay^2 + aay + 2a^3}{-ay}$, or $x = \frac{y^3 + 2ay^2 - aay - 2a^3}{ay}$. I take $y = 0$, then it will be $x = -\frac{2aa}{0} = -\infty$; therefore MA, produced infinitely on the ſide of the negatives, will be alſo an aſymptote to the curve. I take $y = \frac{1}{2}a$, it will be $x = -\frac{15}{4}a$; I take $y = a$, then it will be $x = 0$, and the curve will paſs through D. I take $y = \infty$, it will be $x = \frac{yy}{a} = \infty$, and the curve above AD will go on *ad infinitum*.

nitum. I take $y = 3a = $ AK, then $x = \frac{40}{3}a = $ KP. I take $y = 2a = $ AN, then it will be $x = 6a = $ NR. Now, becaufe, drawing the right line DP, it will be NT $= \frac{40}{6}a$, and $\frac{40}{6}a > 6a$; therefore NT > NR, and the curve in R is convex to the axis AK, that is, concave to the axis AM. But, if it go on towards the afymptote AV, below AK, it muft therefore neceffarily be convex to it, and therefore will have a contrary flexure; but, to determine this does not belong to this place.

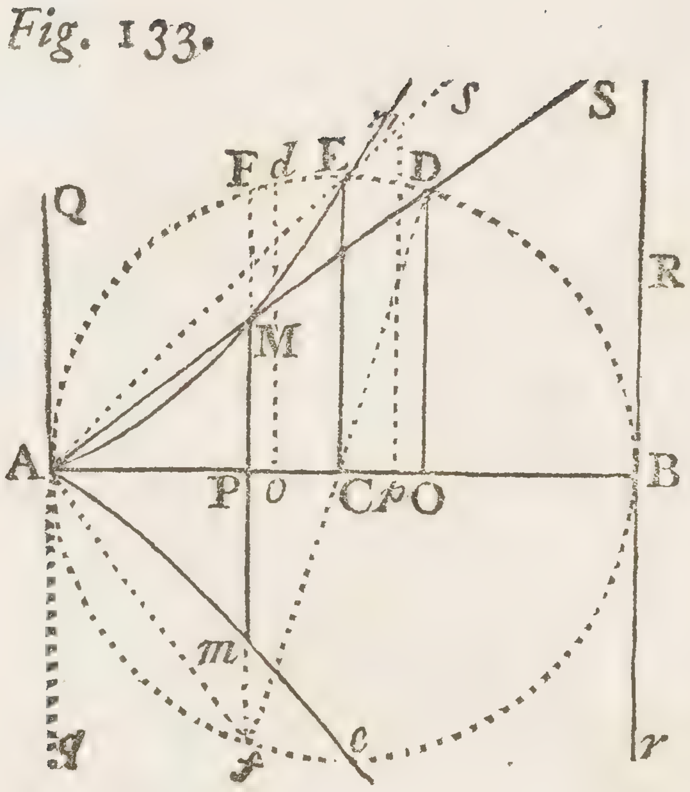
234. But, if the propofed equation of the curve to be conftructed fhall involve both the indeterminates raifed to a power higher than the fecond, fo that it cannot generally be reduced in fuch manner, as that it may have one of the two indeterminates alone, on one fide of the equation, of one power only; then, indeed, the trouble of the operation may increafe, but not the difficulty of the method. For, fixing a known value upon one of the indeterminates, for example x, we fhall have a folid equation, given by y and conftant quantities, which is to be refolved or conftructed; from whence we fhall have the values of y, which will determine fo many points of the curve. Then, fixing upon another value for x, we fhall have another folid equation to be refolved or conftructed, which will furnifh us with other points of the curve; and thus working from one to another fucceffively, we may find as many points as we pleafe of the curve to be defcribed. It may be done by taking points.

235. But, on this and fuch other occafions, as it is required to refolve and conftruct folid equations, as in the fixth Example, it may feem as if we fell into what logicians call *Circulus Vitiofus*, becaufe, in treating of Solid Problems, I have fuppofed the defcription of curves which are fuperior to conic fections. But, upon further reflection, the matter will be found to be much otherwife. For, if the curve to be defcribed be of three or four dimenfions, the folid equation to be conftructed will be of the third or fourth order at moft, and be performed by means of the conic fections. Therefore, without any *circulus vitiofus*, any curve of three or four dimenfions may be defcribed. If the equation of the curve to be defcribed fhall be of five dimenfions, the folid equation to be conftructed will be, at moft, of five; and this is done by means of a curve of three, and one of two dimenfions. And fo, in like manner, of the higher orders; whence it plainly appears, that there can be no objection of our falling into fuch a fallacy. An objection obviated.

PROBLEM I.

Example, for determining the forms of the *loci* from the equation.

Fig. 133.



236. Having given the femicircle AEB, it is required to find the *locus* of the points M fuch, that, if through every one of them a right line be drawn from the extremity of the diameter A, which fhall cut the periphery in D, and if the lines MP, DO, be let fall perpendicular to the diameter, the intercepted lines from the centre CP, CO, may be always equal.

Let M be one of thofe points, and make AB $= a$, AP $= x$, PM $= y$; and, becaufe it muft be CP $=$ CO, it will be OB $=$ AP $= x$, and OD $= \sqrt{ax - xx}$. And, becaufe of fimilar triangles APM, AOD, it will be $x \,.\, y :: a - x \,.\, \sqrt{ax - xx}$, and therefore $y = \frac{x\sqrt{ax - xx}}{a - x}$, that is, $y = \frac{x\sqrt{x}}{\sqrt{a - x}}$, or $y = \frac{xx}{\sqrt{ax - xx}}$, the equation of the curve to be defcribed, which is the *Ciffoid of Diocles.*

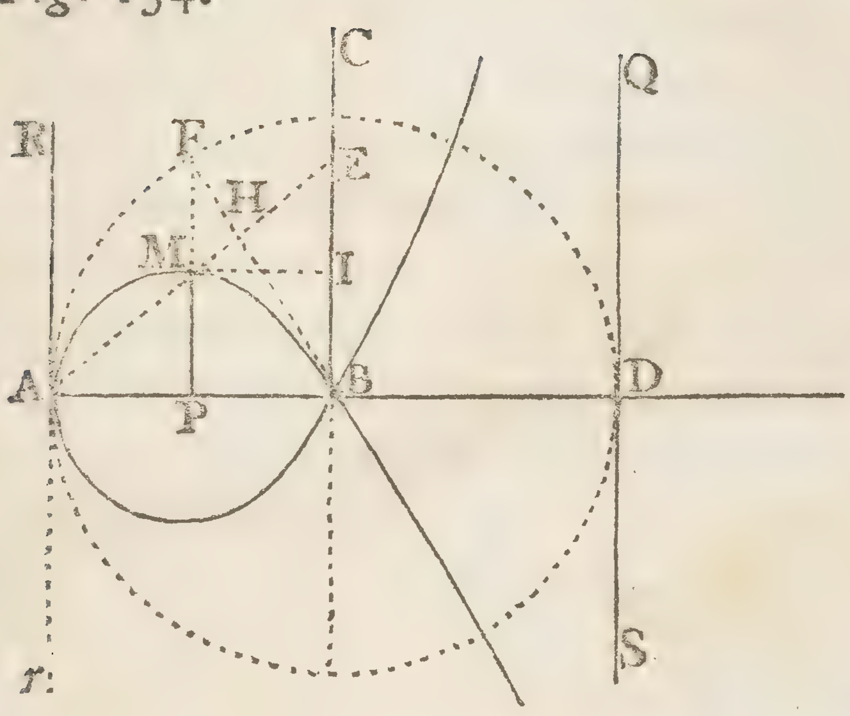
To defcribe it upon the given figure by various points, it may be obferved that the right line AB is the axis of the x's, and A is the given point from whence they take their origin. And, becaufe the y's are perpendicular to this axis, from the point A drawing the tangent AQ, this will be the axis to which the ordinates y are to be referred. Thefe things being premifed, if we make, firft, $x = 0$, to fee if the curve cuts the axis AQ; and, becaufe we find alfo $y = 0$, therefore A will be a point in the curve to be defcribed. Make $y = 0$, to fee if the curve cuts the axis in any other point. But, becaufe we find $x = 0$, the curve will not meet the two axes in any other point but A.

Make $x = \frac{1}{3}a$, it will be $y = \frac{a}{3\sqrt{2}}$; make $x = \frac{1}{2}a$, it will be $y = \frac{1}{2}a$, and therefore, from the centre drawing CE perpendicular to the diameter AB, the curve will pafs through the point E. Make $x = \frac{2}{3}a$, then $y = \frac{4a}{3\sqrt{2}}$; and, laftly, making $x = a$, we fhall find $y = \frac{aa}{0} = \infty$, and therefore the tangent BR to the circle will be the afymptote to the curve. Taking x greater than a, the quantity under the radical fign in the denominator will be negative, and the curve imaginary. Which being alfo imaginary, if we take x negative, it will be

be wholly comprehended between the two tangents AQ, BR, produced *in infinitum*. And, becaufe it approaches to the afymptote BR, having no contrary flexure, it will neceffarily be wholly convex to the axis AB, and will appear as in Fig. 133.

PROBLEM II.

Fig. 134.



237. The angle ABC being a right angle, and the point A in the fide AB being given, the *locus* is required of all the points M, fuch that, drawing through every one of them the right lines AE, terminated at the fide BC in the point E, it may be always EM = EB. Another example for the fame purpofe.

Let any right line AE be drawn, and let M be one of the points required; from the point M let fall MP perpendicular to AB, and make $AP = x$, $PM = y$, and $AB = a$. It will be $PB = a - x$, and $AM = \sqrt{xx + yy}$. Now, becaufe of fimilar triangles APM, ABE, it will be $x \,.\, y :: a \,.\, BE$, and therefore $BE = EM = \frac{ay}{x}$. But it is alfo AP . PB :: AM . ME; that is, $x \,.\, a - x :: \sqrt{xx + yy} \,.\, \frac{ay}{x}$. Therefore $ay = \overline{a - x} \times \sqrt{xx + yy}$, and fquaring, $aayy = aaxx - 2ax^3 + x^4 + aayy - 2axyy + xxyy$, or $\frac{aaxx - 2ax^3 + x^4}{2ax - xx} = yy$. And, laftly, fince the root of $aaxx - 2ax^3 + x^4$ is as well $ax - xx$ as $xx - ax$, it will be $y = \frac{ax - xx}{\sqrt{2ax - xx}}$, and $y = \frac{xx - ax}{\sqrt{2ax - xx}}$: that is, $\pm y = \frac{ax - xx}{\sqrt{2ax - xx}}$, the equation to the curve which is required.

The ordinates y will therefore be pofitive and negative, and equal to each other; and the pofitive and negative will correfpond to the fame abfcifs; and therefore the curve will be both above and below the axis AB, and will be altogether fimilar and equal.

From the point A drawing AR perpendicular to AB, which fhall be the axis to which the ordinates y are referred, as AB is the axis of the abfcifs x; firft, I make

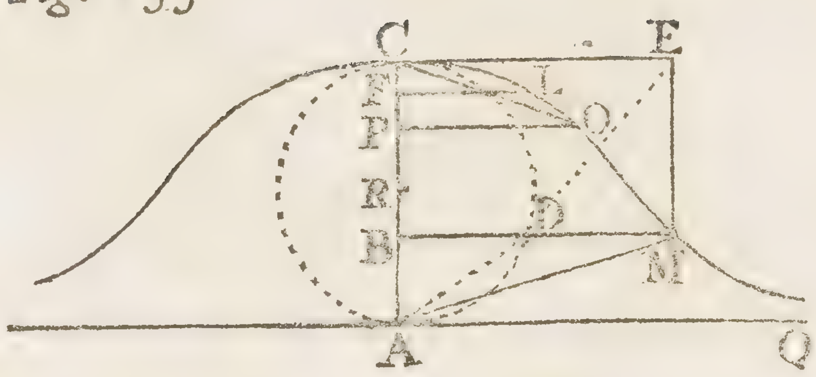
make $x = 0$, to fee if the curve paffes through A; and, becaufe I find alfo $y = 0$, the point A will be the vertex of the curve. Now make $y = 0$, it will be $ax - xx = 0$, and therefore $x = 0$, and $x = a$. Hence I find that the curve will pafs through the point B alfo. Make $x = \frac{1}{3}a$, and it will be $\pm y = \frac{2a}{3\sqrt{5}}$. Make $x = \frac{1}{2}a$, and it will be $\pm y = \frac{a}{2\sqrt{3}}$. Make $x = \frac{4}{3}a$, it will be $\pm y = \frac{4a}{3\sqrt{3}}$. Make $x = 2a$, and it will be $\pm y = \frac{2aa}{0} = \infty$; and therefore, taking $AD = 2a$, and drawing the indefinite right line SQ parallel to PM, it will be an afymptote to the curve. If x be greater than $2a$, the quantity under the radical vinculum will be negative, and therefore the ordinate y will be imaginary, fo that there is no part of the curve beyond the point D. It is plain that, between the points A and B, the curve will be concave towards the axis AB. And becaufe, beyond the point B, it applies itfelf to it's afymptote SQ, it will be convex to the axis BD between B and D, provided it has no contrary flexure.

Taking x negative, the quantity under the vinculum will be always negative, and therefore the ordinate y is imaginary; fo that, on the negative part of the abfcifs, there will be no curve; whence it will be nearly as in Fig. 134.

PROBLEM III.

Another example of the curve called the Witch.

Fig. 135.



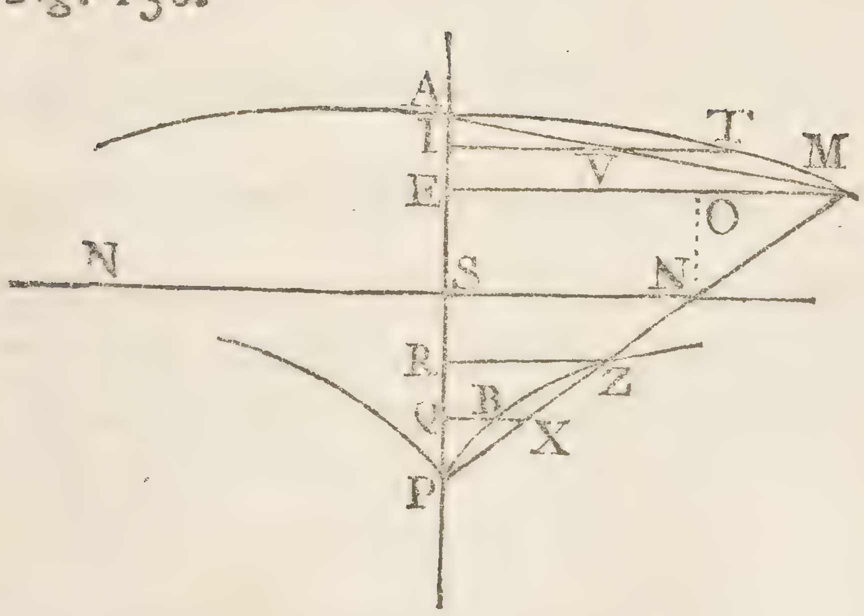
238. The femicircle ADC, on the diameter AC, being given; out of it a point M is required, fuch that, drawing MB perpendicular to the diameter AC, which fhall cut the circle in D, it may be AB . BD :: AC . BM. And, becaufe there will be an infinite number of points that will fatisfy the Problem, the *locus* of thofe points is required.

Let M be one fuch point, and making $AC = a$, $AB = x$, and $BM = y$, by the property of the circle, it will be $BD = \sqrt{ax - xx}$; and, by the condition of the Problem, it is AB . BD :: AC . BM; that is, $x . \sqrt{ax - xx}$ :: $a . y$, and therefore $y = \frac{a\sqrt{ax - xx}}{x}$, or $y = \frac{a\sqrt{a - x}}{\sqrt{x}}$, will be the equation of the curve to be defcribed, which is vulgarly called the *Witch*.

Becauſe AB $= x$, BM $= y$, the axis of the x's will be AC; and AQ parallel to BM, will be the axis of the ordinates y. Firſt, make $x = 0$, it will be $y = \infty$, and therefore AQ is the aſymptote of the curve. Make $y = 0$, it will be $a\sqrt{a - x} = 0$, and therefore $x = a$. So that, when it is $x = a$, the curve will cut the axis AC, and conſequently will paſs through the point C, which will be it's vertex. Make $x =$ AR $= \frac{1}{2}a$, it will be $y = a$. Make $x =$ AP $= \frac{3}{4}a$, it will be $y = a\sqrt{\frac{1}{3}}$. Make $x =$ AF $= \frac{4}{5}a$, it will be $y = a\sqrt{\frac{1}{4}} = \frac{1}{2}a$. Putting x greater than a, the quantity under the vinculum will be negative, and the curve imaginary. To ſee whether the curve be concave or convex towards the axis AC, make this proportion. As CP $= \frac{1}{4}a$ (which correſponds to $x = \frac{3}{4}a$,) is to $y = a\sqrt{\frac{1}{3}}$, ſo is CF $= \frac{1}{5}a$, (which correſponds to $x = \frac{4}{5}a$,) to a fourth, which will be $a \times \frac{4}{5}\sqrt{\frac{1}{3}}$. But $x = \frac{4}{5}a$ gives $y = a\sqrt{\frac{1}{4}}$, and $a \times \frac{4}{5}\sqrt{\frac{1}{3}}$ is leſs than $a\sqrt{\frac{1}{4}}$. Therefore the curve will be concave towards the axis AC. But, becauſe of the aſymptote AQ, it ought alſo to be convex; therefore it will be partly concave and partly convex, and therefore it will have a contrary flexure, which will be found by the method to be given in it's proper place. And taking x negative, becauſe the quantity under the vinculum will be negative in the denominator, y will be imaginary. Wherefore the curve will be as may be ſeen in Fig. 135, obſerving that this curve has a branch ſimilar and equal to the branch CLM, on the other ſide of AC, correſponding to y negative.

PROBLEM IV.

Fig. 136.



239. The indefinite right line NN being given in poſition, and a point P out of the ſame, the point M is required, ſuch that, drawing from it to the point P the right line MP, the line NM, intercepted between the indefinite line NN and the point M, may be equal to a given right line. And, becauſe there are infinite points that ſatisfy this demand, the *locus* of theſe points is required. Another example, being the *Conchoid* of *Nicomedes*.

From the point P draw the right line PA perpendicular to NN, and the right line PM to any point M, which is one of thoſe required; and drawing the right line ME parallel to NN, make PS $= b$, SE $= x$, EM $= y$, and let SA $= a$ be the given line, to which the right line NM is to be equal by the condition of the Problem. From the point N draw the right line NO perpendicular to EM, and it will be MO $= \sqrt{aa - xx}$. And, becauſe of the ſimilar

ſimilar triangles PEM, NOM, it will be PE . EM :: NO . OM, that is $b + x \,.\, y :: x \,.\, \sqrt{aa - xx}$; and therefore $\overline{b + x} \times \sqrt{aa - xx} = xy$, and ſquaring, $xxyy = aaxx - x^4 + 2aabx - 2bx^3 + aabb - bbxx$; and laſtly, $y = \pm \frac{\sqrt{aaxx - x^4 + 2aabx - 2bx^3 + aabb - bbxx}}{x}$, the equation of the curve to be deſcribed, which is the *Conchoid* of *Nicomedes*.

Three different caſes may be diſtinguiſhed in this Problem. That is, it may be $b = a$; it may be b leſs than a; and laſtly, it may be b greater than a. Firſt, let it be $b = a$, and the equation will be changed into this following: $y = \pm \frac{\sqrt{-x^4 + 2a^3x - 2ax^3 + a^4}}{x}$.

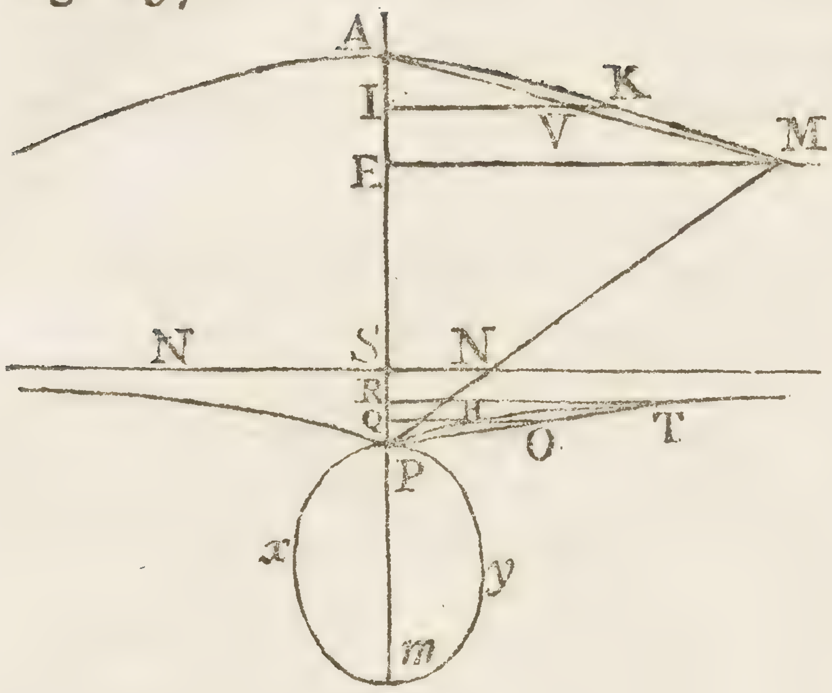
Since it is SE $= x$, and EM $= y$, the axis will be NN, to which the y's are referred, and PA that of the x's, the origin of which is at S. Firſt, I make $x = 0$, to ſee if the curve paſſes through the point S; and becauſe there ariſes $y = \pm \frac{aa}{0}$, that is, y poſitive and negative is infinite, NN will be the aſymptote of the curve. I make $y = 0$, to ſee where the curve cuts the axis PA, and it will be $-x^4 + 2a^3x - 2ax^3 + a^4 = 0$. Now, this equation being reſolved by the rules before taught, it's roots will determine the points in which the curve meets the aforeſaid axis PA. Now the roots of this equation are four, that is, $x = a$ poſitive, and three negative roots equal to it, or $x = -a$. Therefore the curve will meet the axis in two points, diſtant from the point S by the quantity a. But, becauſe, at preſent, we are concerned only with the poſitive x's, it will be ſufficient to conſider the poſitive root; and therefore the curve will paſs through the point A, it being SA $= a$, as is ſuppoſed. Make $x = \frac{1}{2}a$, it will be $y = \pm \frac{\sqrt{27aa}}{2}$. Make $x = \frac{2}{3}a$, then $y = \pm \frac{\sqrt{125aa}}{6}$. Let x be greater than a, and the quantity under the vinculum will be negative, the firſt term, on this ſuppoſition, being greater than the fourth, and the third than the ſecond. Wherefore, taking x greater than a, the curve will be imaginary. It remains to examine whether the curve be always convex towards the axis PA; for it muſt be ſo in part, becauſe of the aſymptote NN. Make then this proportion: As AE $= \frac{1}{2}a$, (which correſponds to $x = \frac{1}{2}a$,) is to $y = \frac{\sqrt{27aa}}{2}$, ſo is AI $= \frac{1}{3}a$ to a fourth, which will be $\frac{\sqrt{27aa}}{3}$. But AI $= \frac{1}{3}a$ correſponds to $x = \frac{2}{3}a$, and therefore to $y = \frac{\sqrt{125aa}}{6}$. Now $\frac{\sqrt{125aa}}{6}$ is greater than $\frac{\sqrt{27aa}}{3}$, and therefore the curve will be partly concave towards the axis PA. Conſequently it will have a contrary flexure,

flexure, as ſhall be ſeen in it's due place. And, becauſe two equal values of y, one poſitive, the other negative, correſpond to the ſame value of x, the curve will have another branch on the negative ſide of y, ſimilar and equal to that on the poſitive ſide; and it will appear as is deſcribed in Fig. 136.

To deſcribe the curve on the negative part of x, it will be neceſſary to change the ſigns of thoſe terms in which the indeterminate is raiſed to an odd power; ſo that the equation will then be $y = \pm \frac{\sqrt{-x^4 - 2a^3x + 2ax^3 + a^4}}{-x}$. Now, firſt, let it be $x = 0$, then $y = \pm \frac{aa}{0}$, and therefore NN is ſtill the aſymptote to the curve on the negative part. Make $y = 0$, and it will be $-x^4 - 2a^3x + 2ax^3 + a^4 = 0$, whence we obtain four roots, as above: three are equal and poſitive, $x = a$, and one negative, $x = -a$. The negative root, which was poſitive in the foregoing caſe, is already fixed in the ſuperior *conchoid*. Then the three equal values ſignify, that, in the pole, which is diſtant from the beginning of the x's by the quantity a, the curve will have a regreſſion, of which we ſhall treat in the Method for Contrary Flexures. Make $x = \frac{1}{2}a$, then $y = \pm \frac{\sqrt{3aa}}{2}$. Make $x = \frac{2}{3}a$, then $y = \pm \frac{\sqrt{5aa}}{6}$. If we take x greater than a, the curve will be imaginary; becauſe, as the quantity under the vinculum is the product of $xx - 2ax + aa$ (a quantity always poſitive,) into $aa - xx$, which, in this ſuppoſition, is negative, the whole quantity under the radical will be negative, and therefore the ordinate y is imaginary. Now, make this proportion: As PR $= \frac{1}{2}a$ (making SR $= \frac{1}{2}a$,) is to $\frac{\sqrt{3aa}}{2}$, ſo is PQ $= \frac{1}{3}a$ (making SQ $= \frac{2}{3}a$,) to a fourth, which will be $\frac{\sqrt{3aa}}{3}$. But $y = \frac{\sqrt{5aa}}{6}$ correſponds to SQ $= \frac{2}{3}a$, or to PQ $= \frac{1}{3}a$, and $\frac{\sqrt{5aa}}{6}$ is leſs than $\frac{\sqrt{3aa}}{3}$; ſo that the curve will be always convex towards the axis NN, ſuppoſing it not to have a contrary flexure; and it will have two equal and ſimilar branches; for two equal values of y correſpond to the ſame x, one of which is poſitive, the other negative. So that the curve will appear as deſcribed in Fig. 136.

240. Now let b be leſs than a; the equation therefore will be $y = \pm \frac{\sqrt{aaxx - x^4 + 2aabx - 2bx^3 + aab0 - bbxx}}{x}$. Make $x = 0$, it will be $y = \pm \frac{ab}{0} = \pm \infty$. Therefore, in this caſe alſo, NN (Fig. 137.) will be the aſymptote Another caſe of the ſame.

Fig. 137.



of the curve. Make $y = 0$, then $a^2x^2 - x^4 + 2a^2bx - 2bx^3 + a^2b^2 - b^2x^2 = 0$; the four roots of which (that is, $x = \pm a$, and two equal to each other, $x = -b$,) will determine the points in which the curve cuts the axis PA. But, at present, it will be enough to consider the positive value $x = a$; and, because SA $= a$, A will be the vertex of the curve. Make $x = \frac{1}{2}a =$ SE, then it will be $y = \pm \frac{\sqrt{3aa + 12ab + 12bb}}{2} =$ EM. Make $x = \frac{2}{3}a =$ SI, then it will be $y = \pm \frac{\sqrt{20aa + 60ab + 45bb}}{6} =$ IK. Make the proportion, AE $= \frac{1}{2}a$ to EM $= \frac{\sqrt{3aa + 12ab + 12bb}}{2}$; so is AI $= \frac{1}{3}a$, to a fourth, which will be $\frac{\sqrt{3aa+12ab+12bb}}{3}$; in order to see if the curve be concave or convex to the axis SA. But, taking AI $= \frac{1}{3}a$, we have SI $= \frac{2}{3}a$, to which corresponds IK $= y = \frac{\sqrt{20aa+60ab+45bb}}{6}$; and it is found to be IV $= \frac{\sqrt{3aa+12ab+12bb}}{3}$ less than IK, or $\frac{\sqrt{20aa+60ab+45bb}}{6}$. Therefore the curve will be concave towards the axis SA. But, as it applies itself continually to the asymptote NN, it will be also convex, and therefore it will have a contrary flexure.

It is plain, that, taking the abscis beyond the point A, that is, x greater than a, there will be no curve; for the second term of the radical will be greater than the first, the fourth greater than the third, and the sixth greater than the fifth; and therefore the quantity under the vinculum will be negative, that is, y will be imaginary.

And, because to the same abscis x two equal ordinates y correspond, one of which is positive, the other negative, the curve on the side of the negative ordinates will also be the same, and nearly as in Fig. 137.

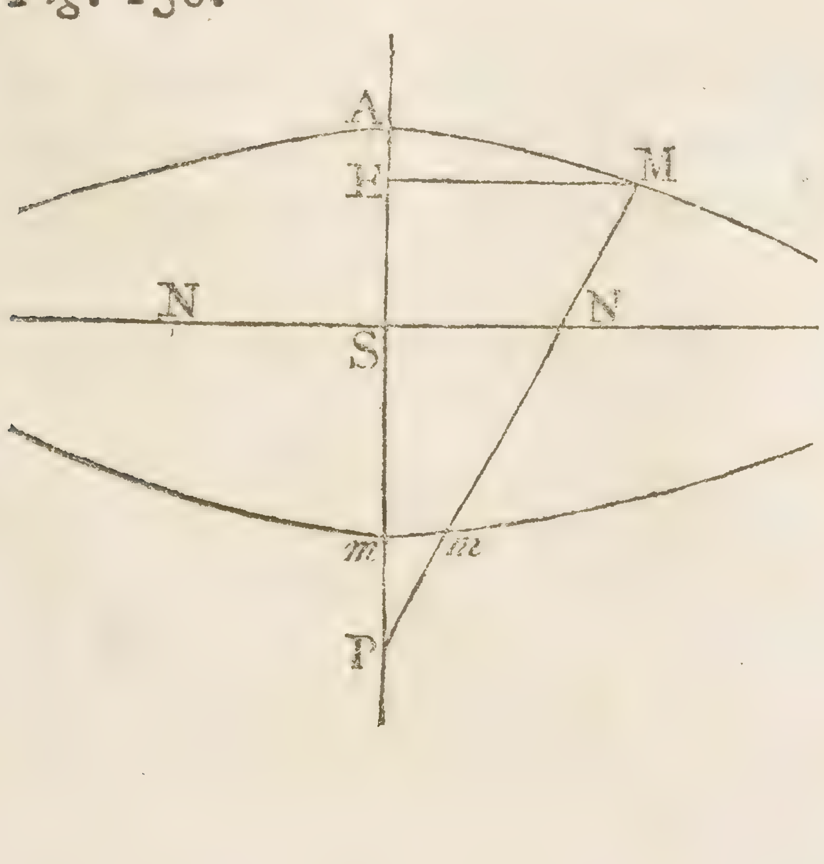
To describe the curve on the side of the abscis x negative, in the equation I change the sign in those terms wherein the power of x is odd, and it is $y = \pm \frac{\sqrt{aaxx - x^4 - 2aabx + 2bx^3 + aabb - bbxx}}{-x}$. Make $x = 0$, then $y = \pm \frac{ab}{0}$, that is, infinite, and therefore NN shall be an asymptote. I make $y = 0$, and it will be $aaxx - x^4 - 2aabx + 2bx^3 + aabb - bbxx = 0$; the four roots

roots of this equation, which are these two, $x = \pm a$, and two equal ones, $x = b$, will determine the points where the curve cuts the axis AP. The negative root $x = -a$ gives me the point A, the positive root $x = a$ the point m, and the two equal roots $x = b$ give the point P, so that there will be a node in the curve. Taking $PR = SR = \frac{1}{2}b = x$, it will be $y = \pm \frac{\sqrt{4aa-bb}}{2} = RT$. Taking $PQ = \frac{1}{3}b$, that is, $SQ = x = \frac{2}{3}b$, it will be $y = \pm \frac{\sqrt{9aa-4bb}}{6} = QH$. I make the analogy, $PR\ (\frac{1}{2}b) \,.\, RT\ (\frac{\sqrt{4aa-bb}}{2}) :: PQ\ (\frac{1}{3}b) \,.\, QO = (\frac{\sqrt{4aa-bb}}{3})$, in order to see whether the curve be concave or convex towards the axis PS. But QO $(\frac{\sqrt{4aa-bb}}{3})$ is greater than QH $(\frac{\sqrt{9aa-4bb}}{6})$; so that the curve is convex towards the axis PS. And this follows also from it's approaching to it as an asymptote.

Taking the abscis beyond the point m, that is, x greater than a, there will be no curve, because the radical aforegoing is the same as $\sqrt{aa-xx} \times \sqrt{x^2-2bx+b^2}$. But, supposing x greater than a, the quantity $aa-xx$ will be negative, and $xx-2bx+bb$ is positive; therefore the product is negative, and the ordinate y is imaginary. Taking the abscis beyond the point P, that is, x greater than b, but less than a, it will be $aa-xx$, a positive quantity, as also, $xx-2bx+bb$; therefore the product is positive, and the ordinate y is real; so that between P and m the curve will correspond, and will form a foliate Pxmy, having a node at P; and the curve will have the appearance nearly as in Fig. 137.

241. Lastly, let b be greater than a; the equation will be the same as in the former case, and, taking the abscis x positive, the curve will be also similar. A third case of the same.

Fig. 138.



Then taking x negative, and supposing $y = 0$, the four roots of the equation, that is, $x = \pm a$, and the two equal roots $x = b$, will give, indeed, the same points, A, m, P, in the axis PA: but the point m will be above the point P. And, assuming the abscis greater than Sm, that is, x greater than a, the quantity $aa-xx$ will be negative; and because $xx-2bx+bb$ is positive, their product will be negative, and therefore the ordinate y will be imaginary. Therefore the curve will not have the foliate

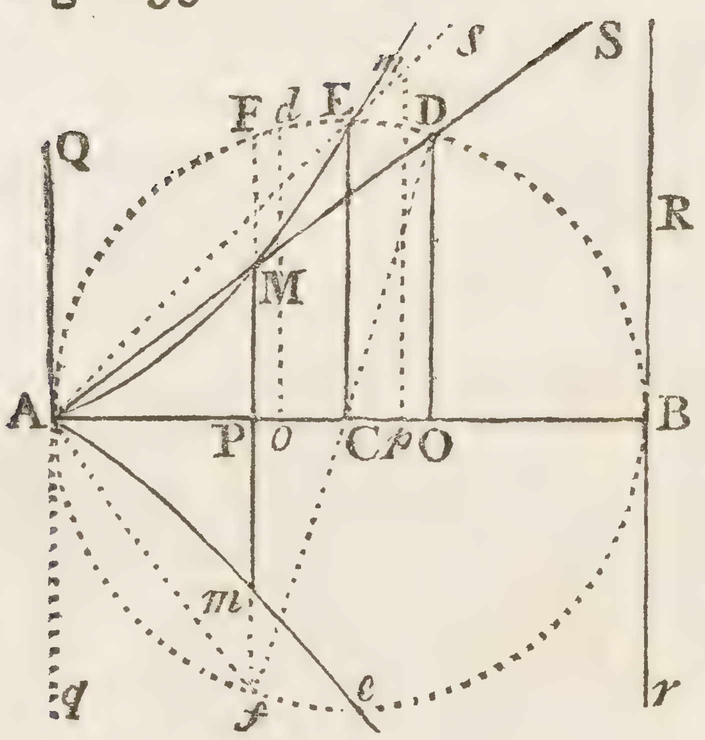
foliate of the former cafe, but will have it's vertex in m. And, becaufe the curve is firft concave, and then convex towards it's axis PS, as is eafily feen, and approaches to the afymptote NN, it will be nearly as in Fig. 138.

The method improved of defcribing curves by points.

242. This method of defcribing curves by an infinite number of points, may perhaps be reduced to a greater perfection, by making ufe alfo of geometrical conftructions. I fhall give fome Examples of it, which may ferve to put the matter in a proper light.

EXAMPLE I.

Fig. 133.

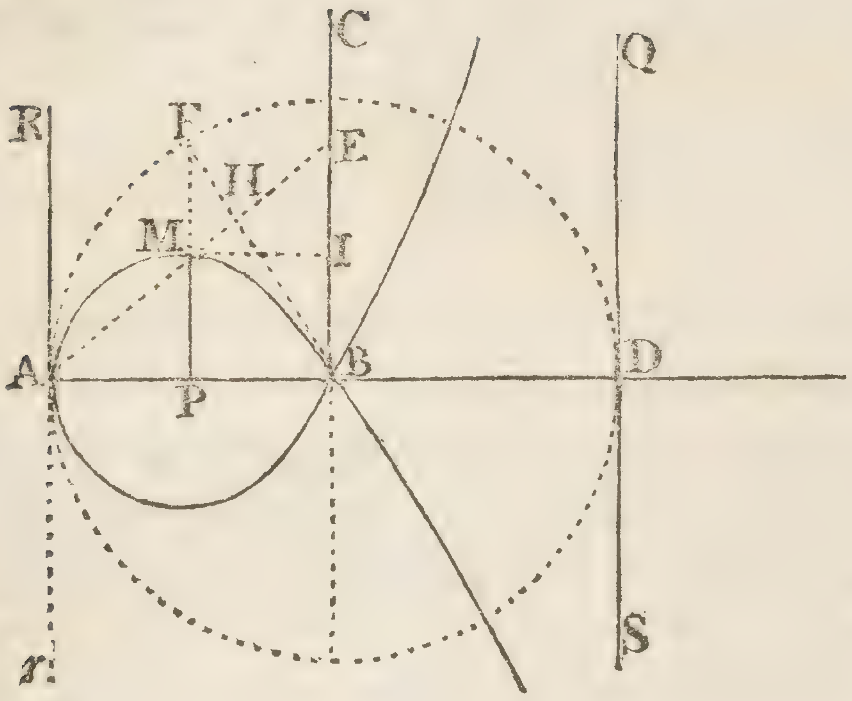


Let us conftruct, by various points, the curve of Prob. I. § 236, which is the *Ciffoid* of *Diocles*, the equation of which was found to be $y = \frac{xx}{\sqrt{ax - xx}}$. With radius AC $= \frac{1}{2}a$ let the circle AEBe be defcribed; and, taking at pleafure AP $= x$, I obferve that the correfponding ordinate Pf is $= \sqrt{ax - xx}$. Through the point f I draw the diameter fCD, and joining the points A, D, with the line AD, the point m, in which it cuts the upper ordinate PF, continued if need be, will be in the *ciffoid*. For, the angle in the femicircle fAD being a right angle, as alfo the angle APM of the co-ordinates, the triangles AfP, APM, will be fimilar, and therefore we fhall have the analogy fP . AP :: AP . PM; that is, $\sqrt{ax - xx} \,.\, x :: x \,.\, y$. Whence it is $y = \frac{xx}{\sqrt{ax - xx}}$. Q. E. I.

After another manner. Becaufe the triangles PCf, CDO, are fimilar, the angles P, O, being right, and the angles at the vertex PCf, DCO, are equal, and alfo Cf = CD, it will be alfo CP = CO, a property of this curve.

EXAMPLE II.

Fig. 134.

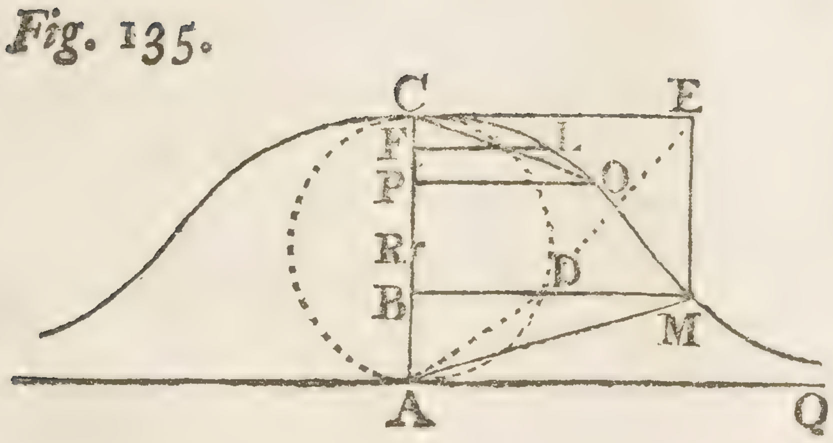


Let the curve be that of Prob. II. § 237, the equation of which is $\pm y = \frac{ax - xx}{\sqrt{2ax - xx}}$. With radius $AB = a$ let the circle AFD be drawn. Taking any line $AP = x$, from the point P draw the ordinate $PF = \sqrt{2ax - xx}$; and drawing the radius BF, let AHE be drawn perpendicular to it. This will cut the ordinate PF, continued if need be, in the point M, which will be in the curve AMB required. For, the triangles AMP, FMH, being fimilar, and likewife the triangles FMH, FBP, the triangle AMP will be fimilar to the triangle BFP, and therefore we fhall have PF . PB :: AP . PM, that is, $\sqrt{2ax - xx} \,.\, a - x :: x \,.\, y$. Whence we have the propofed equation $y = \frac{ax - xx}{\sqrt{2ax - xx}}$. Q. E. I.

After another manner. Becaufe the triangle AMP is fimilar to the triangle AHB; and it has been feen above, that the triangle AMP is alfo fimilar to the triangle FPB. But the fide $AB = BF$; therefore it will be alfo $BH = BP$. Let the right line MI be drawn parallel to AB, and then the triangles BHE, MIE, will be fimilar. But they will be alfo equilateral to each other, it being $BH = BP = MI$. Therefore it will be $EB = EM$, which is the fundamental property of the curve propofed.

EXAMPLE III.

Fig. 135.

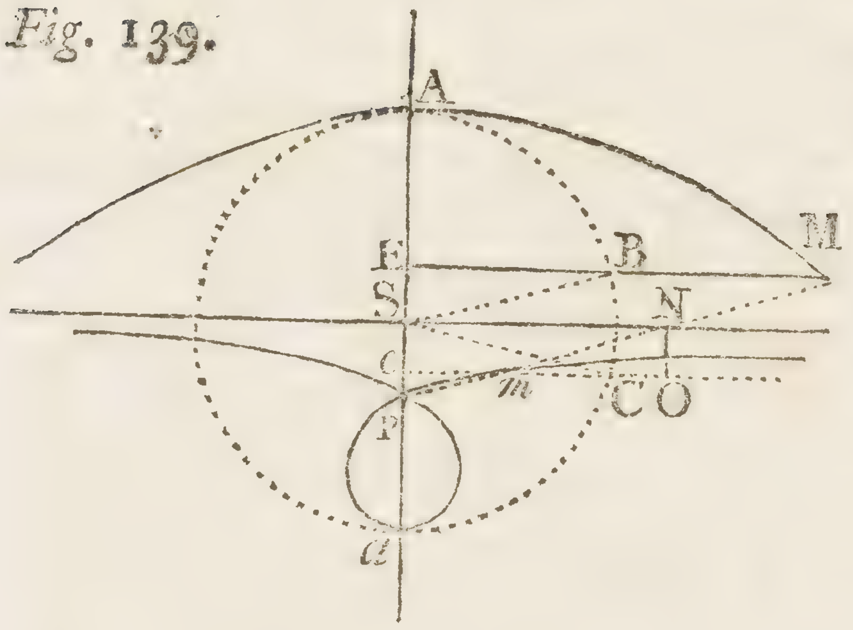


Let the curve to be defcribed be that of Prob. III. § 238, called the *Witch*, the equation of which is $y = \frac{a\sqrt{ax - xx}}{x}$, the diameter of the circle, being $AC = a$. Take any line at pleafure, $AB = x$, and draw

draw the indefinite lines BM, CE, perpendicular to AC. Then through the point D, in which BM cuts the circle, let AD be drawn, which, produced, ſhall cut CE in E. Through the point E draw a parallel to AC; it ſhall meet BM in the point M, which will belong to the curve. For, by the property of the circle, it is BD $= \sqrt{ax - xx}$, and, by ſimilar triangles ABD, ACE, it is AB . BD :: AC . CE. That is, $x \cdot \sqrt{ax - xx} :: a \cdot \text{CE} = \frac{a\sqrt{ax - xx}}{x} = y$, the equation to the given curve.

EXAMPLE IV.

Fig. 139.



Let the *Conchoid* of *Nicomedes* of Prob. IV. § 239, be to be deſcribed by various points. It's equation is $\pm y = \frac{\overline{b \pm x} \times \sqrt{aa - xx}}{\pm x}$. Make SA $=$ Sa $= a$, SP $= b$. With radius SA $= a$, let there be deſcribed the circle ABCa, and taking at pleaſure two abſciſſes SE, Se, equal to each other, which may be called x poſitive and negative, draw the ordinates EB, eC, each of which ſhall be $= \sqrt{aa - xx}$, and let them be produced indefinitely beyond the points B, C. Through the points S, B, let the right line SB be drawn, and through the point P a parallel to it, PM. The two points M, m, in which PM cuts the two right lines EB, eC, ſhall belong to the curve required; that is to ſay, the point M to the ſuperior branch, and m to the inferior branch of the *conchoid*.

And as to the point M; becauſe the two triangles SEB, PEM, are ſimilar, it will be SE . EB :: PE . EM; that is, $x \cdot \sqrt{aa - xx} :: b + x \cdot y$. And conſequently the equation will be $y = \frac{\overline{b + x} \times \sqrt{aa - xx}}{x}$, in reſpect of the upper branch of the *conchoid*.

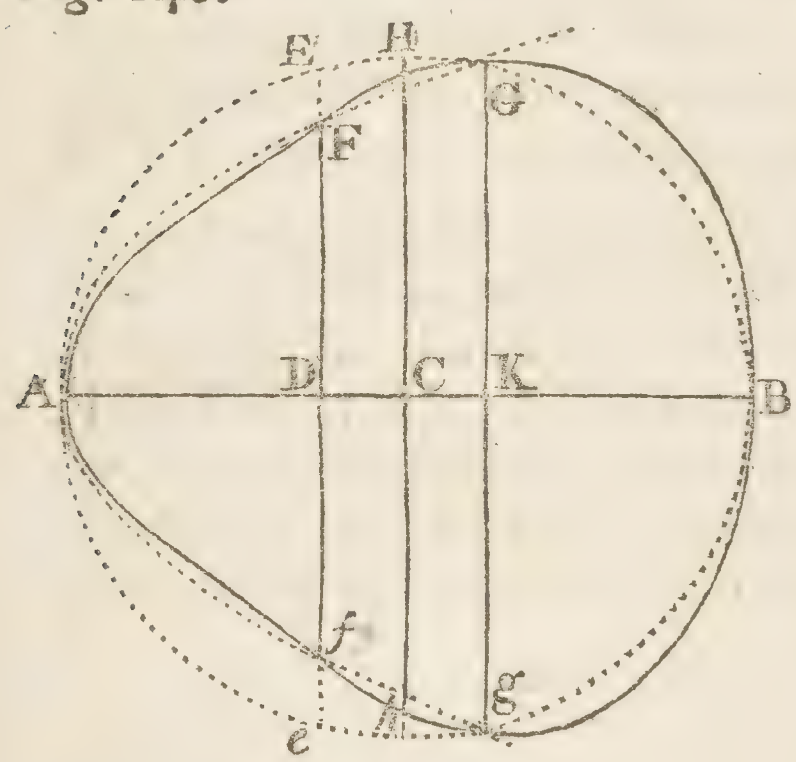
Then, as to the point m; drawing the line SC, the triangle SeC will be ſimilar and equal to the triangle SEB. For the triangle Pem is ſimilar to the triangle SEB; therefore alſo it will be ſimilar to SeC, and therefore we ſhall have the analogy Pe . em :: Se . eC; that is, $-x \cdot \sqrt{aa - xx} :: b - x \cdot y$. Whence we have the equation $y = \frac{\overline{b - x} \times \sqrt{aa - xx}}{-x}$; which is the very ſame as ſhould belong to the lower branch of the curve.

Through

Through the point S drawing the indefinite line SN parallel to the ordinates EM, *em*, from the construction above we shall easily obtain the principal property of the *conchoid*; which is, that, from the point or pole P, if we draw PM cutting the curve in the points M, *m*, and the line SN in the point N, the intercepted lines *m*N, NM, between the curve and the indefinite line SN, will always be of a constant length, and equal to SA $=$ SB $= a$. For, by the construction, SBMN will be a parallelogram, and therefore NM $=$ SB. But, drawing NO parallel to S*e*, the triangles SBE, *m*NO, will be similar; and besides, NO $=$ S*e* $=$ SE. Therefore it will be *m*N $=$ SB, and consequently *m*N $=$ NM. Q. E. D.

243. The constructions of the three first Examples come out pretty simple, there being nothing required to be done, but to draw a circle with a given diameter, and some right lines. On other occasions the Conic Sections must be admitted, which are sometimes to be described with variable diameters, parameters, and rectangles. But these may be taken as constant, in determining one or more points of the curve. —Improved by the conic sections.

Fig. 140.



To give an example of it. Let us construct, by points, the curve belonging to this equation $x\sqrt{2ax - xx} = yy$. Draw the circle AHB*b*, whose diameter is AB $= 2a$. Take at pleasure AD $=$ KB $= x$; it will be DE $=$ KG $= \sqrt{2ax - xx}$. With parameter DE, to the axis AB, describe the *Apollonian* parabola GFA*fg*, and DF, D*f*, will give the positive and negative values of y, making $x =$ AD. And KG, K*g*, the positive and negative values of y, making $x =$ AK. Wherefore the four points F, *f*, G, *g*, will be in the curve required. By a like method, and by varying the value of y, we may determine other points of the curve.

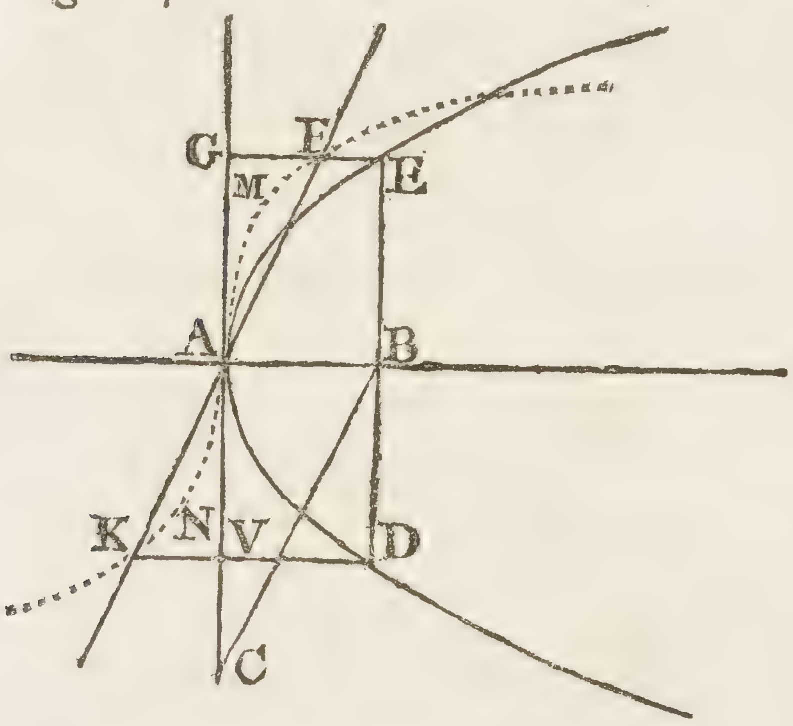
244. A second manner of constructing curves beyond the second degree, will be that mentioned at § 220, by means of other lines of a lower degree. And, to begin with parabolas of any degree, it may be first observed, that the *Apollonian* parabola is the only one of it's kind, and is expressed by the equation $ax = yy$. The cubic parabolas are two, that is, $aax = y^3$, and $axx = y^3$. Those of the fourth degree are three, $a^3x = y^4$, $aaxx = y^4$, and $ax^3 = y^4$. And, in general, those of the degree expressed by n are in number $n - 1$, and are $ax^{n-1} = y^n$, $aax^{n-2} = y^n$, $a^3x^{n-3} = y^n$, $a^4x^{n-4} = y^n$, and so on successively, till the exponent of x is unity. —By parabolas of higher degrees.

The first cubical parabola constructed.

245. All those which have x, with unity, for it's exponent, are called first parabolas. Thus, $aax = y^3$, $a^3x = y^4$, $a^{n-1}x = y^n$, are all first parabolas.

To construct any parabola of any degree whatever, the beginning must be from the first cubic parabola $aax = y^3$.

Fig. 141.



It is plain that this must have two branches, one positive, the other negative; for, taking x positive, y will also be positive, that is, $y = \sqrt[3]{aax}$, and this will be it's positive branch. But, taking x negative, y will also be negative, or $y = \sqrt[3]{-aax}$, (which is no imaginary quantity,) and this will be the negative branch. It is evident that these two branches go on *ad infinitum*, and are concave to the axis AB.

To proceed to the construction. Make $yy = az$; and, substituting in the equation $aax = y^3$ this value of yy, the equation to the cubic parabola will be changed into this, $ax = zy$, which may be resolved into the following analogy, $a \,.\, z :: y \,.\, x$.

This supposed, let the parabola of the equation $yy = az$ be described to the axis AB, and let it be DAE. Make $AB = z$, $BE = y$, $BD = -y$, $AC = a$. Draw CB, and through the point A draw the line KAF parallel to CB; and making AG = BE, draw GE. It will be CA . AB :: AG . GF, that is, $a \,.\, z :: y \,.\, x$. Whence taking AB at pleasure, the corresponding lines BE, or AG, and GF, will be the co-ordinates of our cubic parabola, and F will be a point of it. For, in the analogy $a \,.\, z :: y \,.\, x$, restoring the value of z, or $\frac{yy}{a}$, it will be $a \,.\, \frac{yy}{a} :: y \,.\, x$, or the equation $y^3 = aax$.

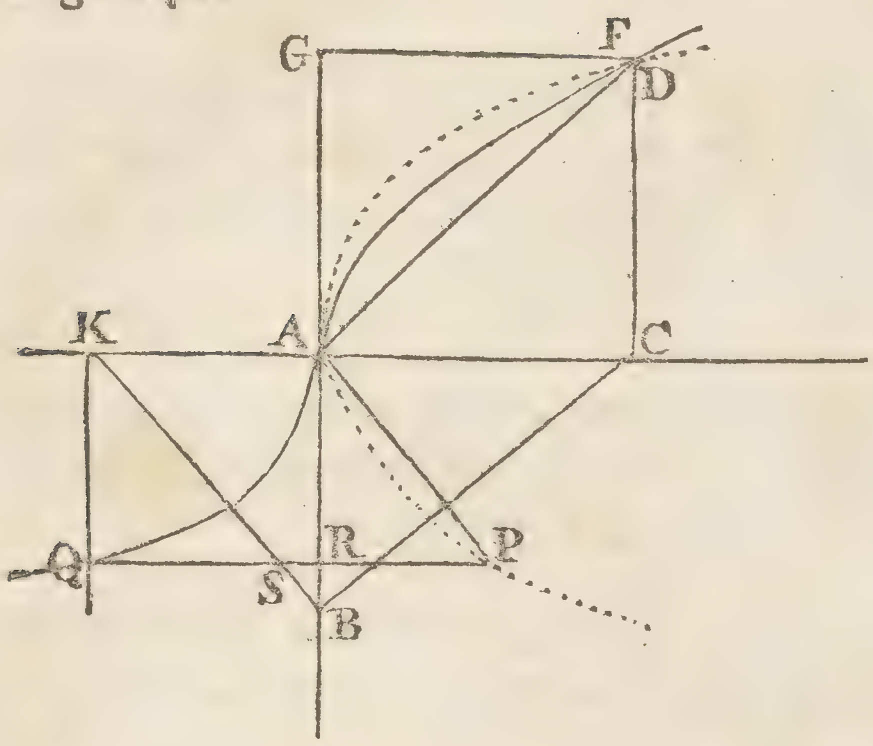
Now, because, when x is taken negative, y will be negative also, the analogy $a \,.\, z :: y \,.\, x$ will be changed into this following, $a \,.\, z :: -y \,.\, -x$; whence, taking AV = BD, it will be CA . AB :: AV . VK; that is, $a \,.\, z :: -y \,.\, -x$, and the point K will be in the cubical parabola. The branch AMF will be positive, and ANK the negative branch.

The first parabola of the fourth degree constructed.

246. Let it be proposed to construct the first parabola of the fourth degree $a^3x = y^4$. This will have also two branches, one above the axis, the other below it, because to x positive corresponds both y and $-y$, for the index of the power of y is an even number. These two branches will be concave towards the axis, and will proceed *in infinitum*. To go on to the construction. I make $y^3 = aaz$, and, instead of y^3, substituting this value in the equation proposed, we shall have $zy = ax$, or $a \,.\, z :: y \,.\, x$.

To

Fig. 142.



To axis KC let the parabola of the equation $y^3 = aaz$ be described, which, because it is the first cubic, we know already how to construct; and let this be QAD. It will be AC = GD = z, AK = $-z$, and CD = AG = y, KQ = $-y$. Take AB = a, and draw the right lines BC, BK, and through the point A draw AF parallel to BC, and AP parallel to KB. This supposed, it will be BA . AC :: AG . GF, that is, $a . z :: y . x$; and the point F will be in the curve-line proposed to be constructed. For, it being $a . z :: y . x$, and $z = \frac{y^3}{aa}$, it will be $a . \frac{y^3}{aa} :: y . x$; that is, $a^3x = y^4$.

But, because when x is positive we may take y negative, which in this case will be KQ, and AK will be $-z$, we should have also BA . AK :: KQ (= AR) . RP; or $a . -z :: -y . x$. Therefore the point P will also be in the curve $a^3x = y^4$.

247. Let it be proposed to construct the first parabola of the fifth degree, $a^4x = y^5$. This will also have two branches, one positive, the other negative. For, taking x positive, y will be positive, that is, $y = \sqrt[5]{a^4x}$. But, taking x negative, y will be negative, that is, $y = \sqrt[5]{-a^4x}$. These two branches go on infinitely, and are concave to the axis AB. To proceed to the construction. Make $y^4 = a^3z$, and substituting this value in the proposed equation, it will be $ax = yz$, or $a . z :: y . x$.

The first parabola of the fifth degree constructed.

To the axis AB (Fig. 141.) describe the parabola of the equation $y^4 = a^3z$, and let it be DAE. It being AB = z, it will be BE = y, and BD = $-y$. Make AC = a, and draw CB, and KAF parallel to it. Then draw the right line EFG, and the parallel DVK. This supposed, it will be CA . AB :: AG . GF, or $a . z :: y . x$; and the point F will be in the curve to be constructed. For, it being $a . z :: y . x$, as also, $a^3z = y^4$, it will be $a . \frac{y^4}{a^3} :: y . x$, or $y^5 = a^4x$, the equation to the curve proposed.

Now, because, x being negative, y will also be negative, the analogy $a . z :: y . x$ will be changed into this, $a . z :: -y . -x$. Wherefore, taking AV = DB, it will be CA . AB :: AV . VK, or $a . z :: -y . -x$. Whence the point K will be in the curve proposed to be constructed. The branch AMF will be positive, and ANK will be the negative branch.

248. And,

The firſt parabola of any degree conſtructed.

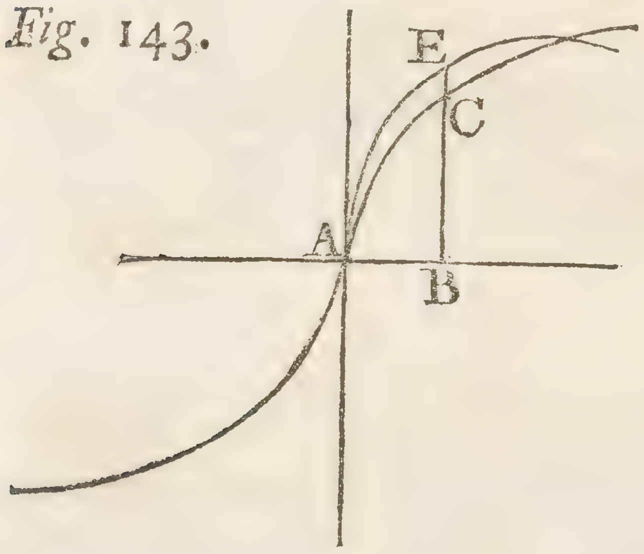
248. And, in general, let it be propoſed to conſtruct the parabola whoſe equation is $a^{n-1}x = y^{n}$. Make $y^{n-1} = a^{n-2}z$, and ſubſtituting this value in the propoſed equation, we ſhall ſtill have $zy = ax$. Whence it may be perceived, that we may always conſtruct any firſt parabola by means of a triangle, and of the firſt parabola of the next inferior degree.

Conſtruction of other ſucceeding parabolas.

249. Now it will be eaſy to go on to the conſtruction of the other ſucceeding parabolas, or thoſe of the ſecond, third, fourth, &c. of any degree; for theſe alſo may be conſtructed by the conſtruction of their firſt parabolas.

Let it be propoſed to conſtruct the ſecond cubic parabola, whoſe equation is $axx = y^3$. I make $y^3 = aaz$, and, by ſubſtituting, inſtead of y^3, it's value in the propoſed equation, it will be $xx = az$.

Fig. 143.



To the axis AB let there be deſcribed the *Apollonian* parabola AC, whoſe equation is $xx = az$; then to the ſame axis deſcribe the firſt cubic parabola of the equation $y^3 = aaz$; and it being AB $= z$, it will be BE $= y$. But, in the *Apollonian* parabola AC, becauſe AB $= z$, it will be BC $= x$. Therefore we ſhall always have the two co-ordinates x, y, of the ſecond cubic parabola.

Let it be propoſed to conſtruct the third parabola of the fourth degree, whoſe equation is $ax^3 = y^4$. I make $a^3z = y^4$, and, by ſubſtitution, it will be $x^3 = aaz$. Let this firſt cubical parabola $x^3 = aaz$ be conſtructed, and to the ſame axis let there alſo be conſtructed the firſt of the fourth degree, $y^4 = a^3z$. The two ordinates of theſe curves, correſponding to the ſame abſciſs z, will give the co-ordinates x, y, of the propoſed curve.

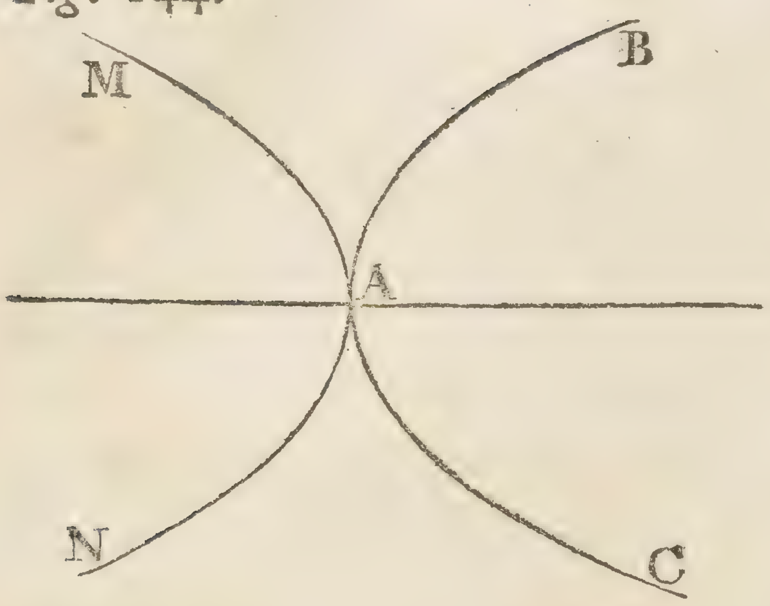
In the conſtruction of all others, of any ſuperior degree, we may proceed in the ſame method; theſe examples are ſufficient, the thing itſelf being very plain.

Squaring the equation produces a reduplication of the curve.

250. It only remains to be obſerved, that the ſecond parabola of the fourth degree, $aaxx = y^4$, is no other than the *Apollonian* parabola, but redoubled the contrary way. For, firſt, if it be $aaxx = y^4$, it will be alſo, by extracting the fourth root, $\sqrt[4]{aaxx} = \sqrt{ax} = \pm y$. But $\sqrt{ax} = \pm y$, or $ax = yy$, is no other than the equation to the *Apollonian* parabola. Our curve is therefore a common parabola, but redoubled; becauſe the term $aaxx$ is alike generated, as well from $+ax \times +ax$, as from $-ax \times -ax$; which may be equally verified, becauſe $\sqrt[4]{aaxx} = \sqrt[4]{\overline{+ax} \times \overline{+ax}} = \sqrt[4]{\overline{-ax} \times \overline{-ax}} = \sqrt{ax} = \pm y$.

Wherefore,

Fig. 144.



Wherefore, to negative x will correſpond real y, and the branch MAN on the negative ſide will be perfectly like the branch BAC on the poſitive ſide, having reſpect to both the expreſſions $\sqrt[4]{aaxx} = \sqrt{ax} = \pm y$. But the *Apollonian* parabola has no branch on the negative ſide; for, putting x negative, it will be $\sqrt{-ax} = \pm y$; ſo that the curve will be imaginary.

If we raiſe the equation $ax = yy$ to the third power, the curve correſponding to the equation $a^3x^3 = y^6$ will be no other than the *Apollonian* parabola only. Raiſing the equation $ax = yy$ to the fourth power, the curve correſponding to the formula $a^4x^4 = y^8$ becomes the common parabola redoubled the contrary way. And, in general, if the power to which the formula $ax = yy$ is raiſed ſhall be even, the *Apollonian* parabola redoubled will exhibit the curve; if the power be odd, the common parabola will be ſufficient.

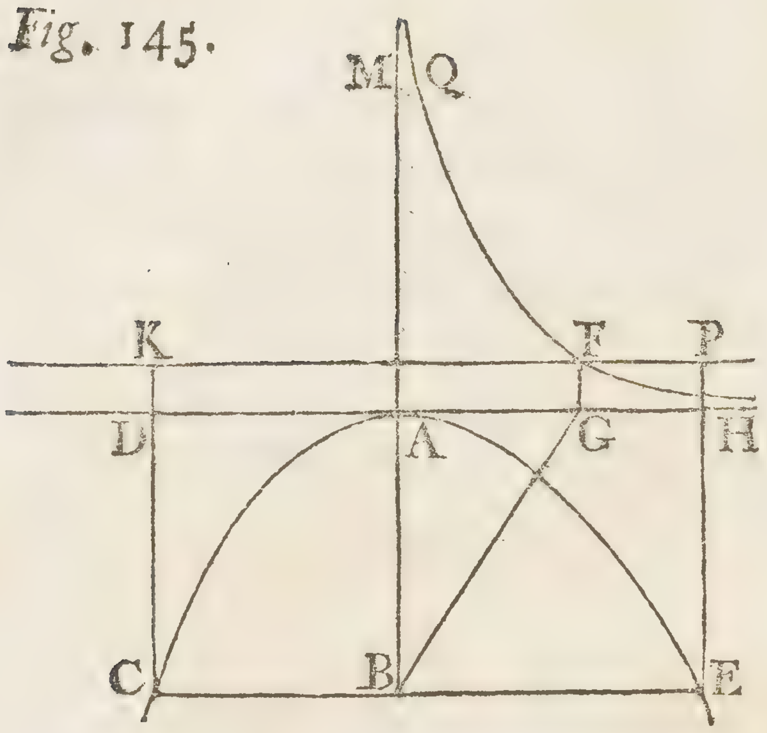
The ſame doctrine may be applied to all firſt parabolas and hyperbolas, whoſe canonical equations are $a^{n-1}x = y^n$, taking for n any integer number, affirmative or negative. This being raiſed to an even power, the proper curve of the new equation will be the parabola or hyperbola $a^{n-1}x = y^n$ redoubled the contrary way. If the power be odd, the reduplication vaniſhes, and there will remain the ſimple genuine curve of the equation $a^{n-1}x = y^n$.

251. From the conſtruction of parabolas of any degree, we may go on to the conſtruction of hyperbolas alſo of any degree. Conſtruction of hyperboloids.

The hyperboloids of the third degree are two; that is, $a^3 = xxy$, and $a^3 = xyy$. Let it be propoſed to conſtruct the hyperboloid of the equation $a^3 = xxy$. This curve will have two branches which approach to aſymptotes; both of them will have their ordinates poſitive, but the abſciſſes in one will be poſitive, in the other negative.

To conſtruct it, make $xx = az$, and, by ſubſtitution, it will be $aa = zy$. Between the aſymptotes AM, AG, (Fig. 145.) deſcribe the hyperbola FQ of the equation $aa = zy$. Then taking AG $= z$, it will be GF $= y$; then from the point G, at half a right angle, let be drawn GB, and it will be AB $=$ AG $= z$. To the axis AB let there be deſcribed the parabola CAE of the

Fig. 145.

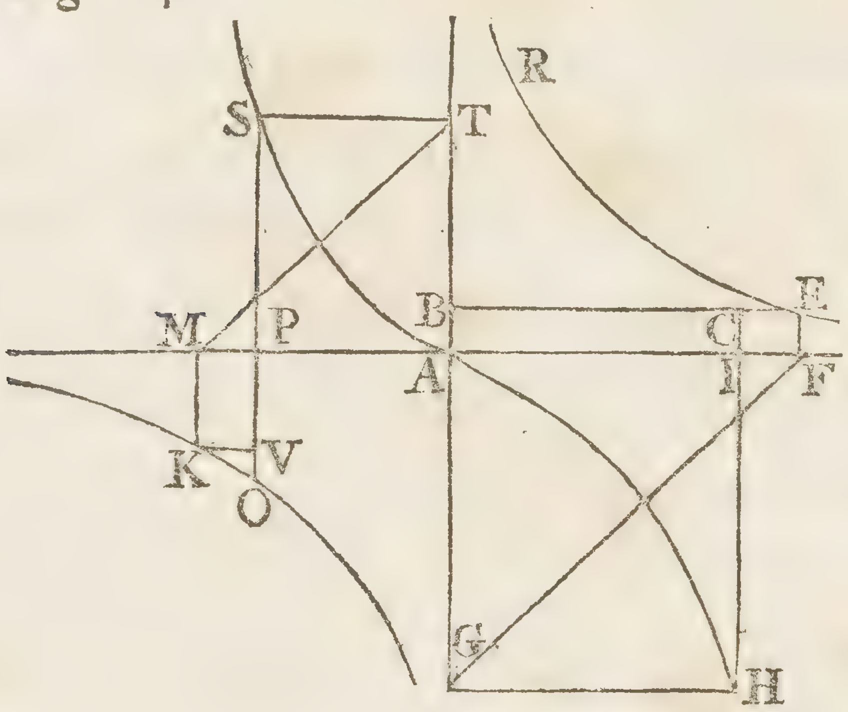


equation $az = xx$, and drawing the ordinates BC, BE, and the indefinite lines CK, EP, parallel to BA, it will be AH = BE $= x$. And drawing FK parallel to GD, it will be HP = GF $= y$. In the fame manner, it will be AD = BC $= -x$, DK $= y$; and the points P, K, will be in the curve propofed.

I forbear to give the conftruction of the equation $a^3 = xyy$, becaufe it is the very fame curve, only the co-ordinates have changed their places.

—of higher hyperboloids. 252. Let there be propofed an hyperboloid of the fourth degree, and let it's equation be $a^4 = x^3y$. This curve will have two branches, which apply to afymptotes, in one of which x will be pofitive, and y pofitive, and in the other x will be negative, and y negative.

Fig. 146.



Put $x^3 = aaz$, and, by fubftitution, we fhall have $zy = aa$. Between the afymptotes MF, TG, produced indefinitely, let the hyperbola of the equation $zy = aa$, or ER, KO, be defcribed. Then it will be AF $= z$, FE $= y$, AM $= -z$, MK $= -y$. From the point F, at half a right angle, draw FG, to which let MT be parallel, and it will be AG = AF $= z$, and AT = AM $= -z$. To the axis TG let be defcribed the cubic parabola SAH of the equation $x^3 = aaz$, and it will be AI = GH $= x$, and AP = TS $= -x$. Whence, drawing the right lines EC, KV, parallel to AI, it will be IC $= y$, and PV $= -y$, and the points C, V, will be in the propofed curve.

Here, alfo, I omit the conftruction of the equation $a^4 = xy^3$, becaufe, only changing the places of the co-ordinates, it is the fame as before. Alfo, I omit the conftruction of the equation $a^4 = xxyy$, becaufe it is reduced to the *Apollonian* hyperbola.

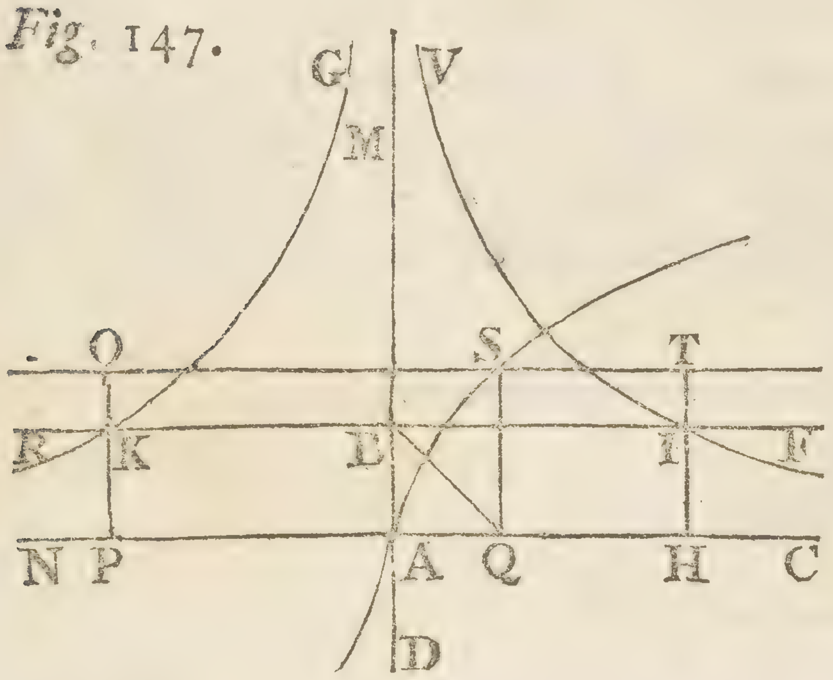
Other hyperboloids conftructed. 253. Let the hyperboloid of the fifth degree be propofed, and, firft, let the equation be $a^5 = x^4y$. This will have two branches, which approach to afymptotes; in one of which, taking x pofitive, y will alfo be pofitive. In the other, taking x negative, yet, however, y will be pofitive.

Make

Make $x^4 = a^3z$; then, ſubſtituting, it will be $aa = zy$. Between the aſymptotes AG, AM, (Fig. 145.) deſcribe the *Apollonian* hyperbola FQ of the equation $aa = zy$; then, taking $AG = z$, it will be $GF = y$. From the point G, at half a right angle, draw the right line GB, and it will be $AB = AG = z$. To the axis AB deſcribe the parabola CAE of the equation $x^4 = a^3z$, and it will be $BE = AH = x$, $BC = AD = -x$; and, drawing FK parallel to GD, and CK, EP, perpendicular to the ſame, it will be $HP = DK = GF = y$, and the points P, K, will be in the curve propoſed.

Let $a^5 = x^3y^2$ be another equation of the hyperboloid of the ſame degree; this will have two branches, becauſe to the ſame poſitive x will correſpond two ordinates y, one poſitive, the other negative.

Fig. 147.



Make $x^3 = aaz$; then, ſubſtituting, it will be $a^3 = zyy$. Between the aſymptotes DM, CN, let there be deſcribed the hyperboloid RG, FV, of the equation $a^3 = zy^2$, and making $AH = y$, $AP = -y$, it will be $HI = z = PK = AB$. To the axis PH let there be deſcribed the cubic parabola AS of the equation $x^3 = aaz$, and from the point B draw BQ at half a right angle, and raiſe the perpendicular QS: then it will be $AQ = z$, $QS = x$. Through the point S draw the right line OT parallel to the aſymptote NC, which may meet the produced lines HI, PK, in the points T, O. Then, it being $AH = y$, it will be $HT = x$, $AP = -y$, $PO = x$; and the points O, T, will be in the curve propoſed.

The conſtructions of the other two equations, $a^5 = x^2y^3$, and $a^5 = xy^4$, will be after the ſame manner, only making the co-ordinates to change places. And by the ſame artifice may all the hyperboloids of any degree be eaſily conſtructed.

254. It may be obſerved, that all the firſt parabolas, which are deſcribed about one and the ſame axis, will cut one another in the ſame point. For, taking for every one of them the ſame abſciſs $x = a$, they will all have the ſame correſponding ordinate $y = a$; which could not be, except they all cut in the ſame point. Obſervation on the forms of the firſt paraboloids.

255. Alſo, the parabolas of higher dimenſions (meaning higher than the firſt,) tend firſt to arrive at the point of ſection, above thoſe of an inferior degree, approaching nearer to the tangent of the vertex, and after the ſection they approach to the axis, theſe more than thoſe. For, in the *Apollonian* parabola, it being $y = \sqrt{ax}$, in the firſt cubic, $y = \sqrt[3]{aax}$, in the firſt of the fourth degree, —of higher paraboloids and hyperboloids.

degree, $y = \sqrt[4]{a^3x}$, and ſo on; if we take x leſs than a, then $\sqrt{ax}$ will be leſs than $\sqrt[3]{aax}$, and this will be leſs than $\sqrt[4]{a^3x}$, and ſo on. But, on the contrary, taking x greater than a, it will be $\sqrt{ax}$ greater than $\sqrt[3]{aax}$, this greater than $\sqrt[4]{a^3x}$; and ſo on.

After the ſame manner, and for a like reaſon, the hyperboloids (meaning alſo the firſt,) all cut one another at the vertex, and thoſe of higher dimenſions tend after the point of ſection between thoſe of lower dimenſions and the aſymptote in which the x's are taken. And on the part of the aſymptote, parallel to y, the inferior tend within, between thoſe of higher dimenſions and the aſymptote.

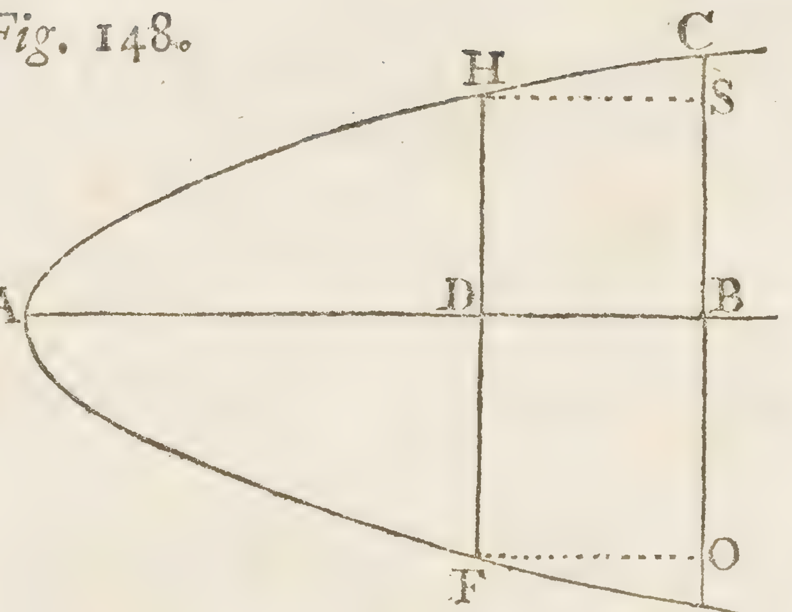
Curves of ſeveral terms conſtructed; divided into three caſes.

256. There remains now to conſtruct ſuch equations as have ſeveral terms, in which I ſhall diſtinguiſh three caſes. Thoſe of the firſt caſe I call ſuch, which have one term only, in which the indeterminate y is found, and that of one dimenſion alone. Of the ſecond caſe are thoſe, which have one term only in which y is found, but that raiſed to any power. Thoſe are of the third caſe which have many terms in which y is found; and that raiſed to any power.

CASE I. EXAMPLE I.

An example of the firſt caſe.

Fig. 148.

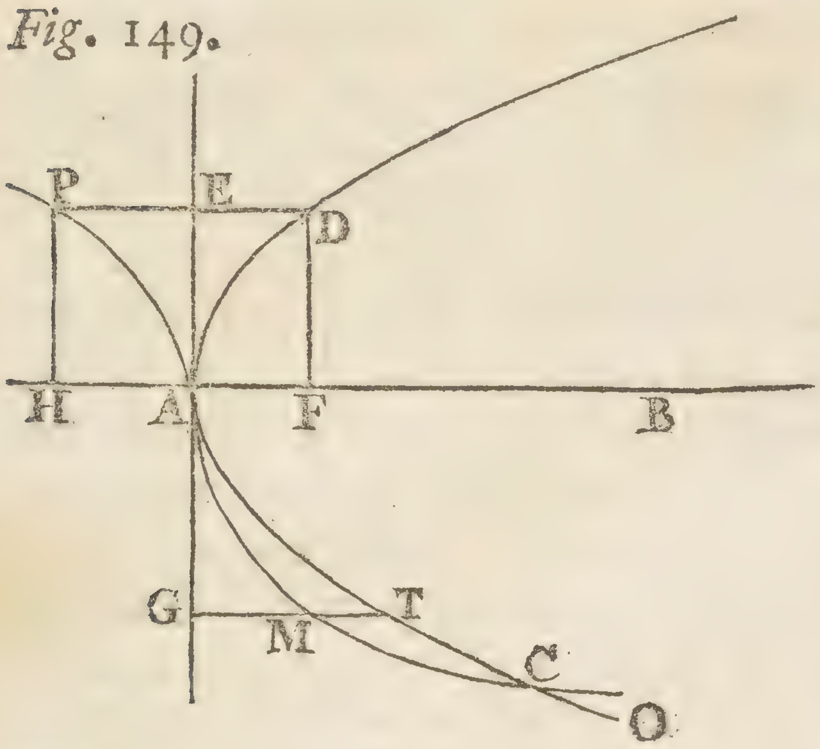


257. Let it be propoſed to conſtruct the curve of this equation $a^4 - x^4 = a^3y$. Make $y = t - q$, by which the given equation may be reſolved into theſe two, $a^4 = a^3t$, $x^4 = a^3q$. To the axis AB let the parabola MAC of the equation $x^4 = a^3q$ be deſcribed; and it being $AD = q$, it will be $DH = x$, $DF = -x$. But, by the equation $a^4 = a^3t$, it is $t = a$; and therefore, taking $AB = a = t$, it will be $t - q = y$. Whence, taking at pleaſure any abſciſs $BS = DH = x$, and $BO = DF = -x$, the lines SH, OF, parallel to BA, will be the correſponding ordinates of the curve propoſed, which is one portion of the ſame parabola of the fourth degree.

EXAMPLE II.

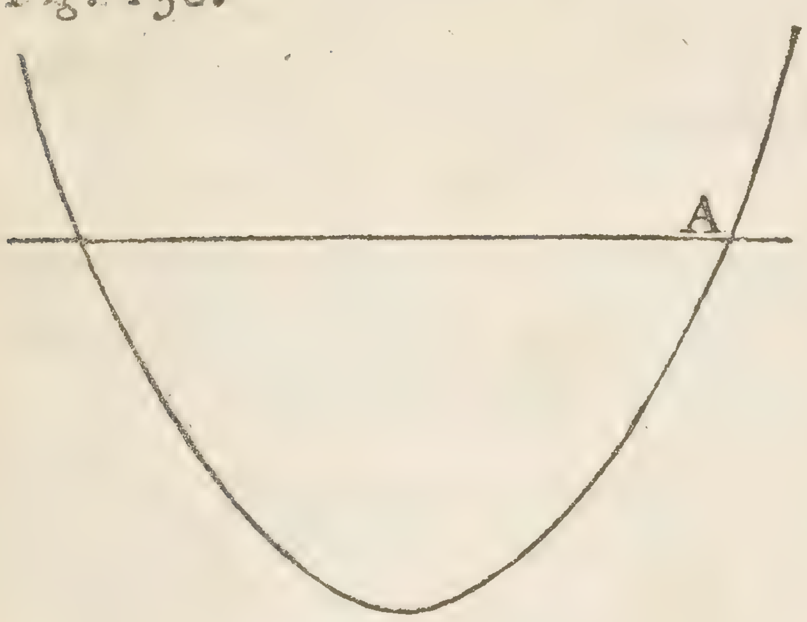
258. Let it be propofed to conftruct the curve of the equation $x^4 + ax^3 = a^3y$. By the rules already known, we may perceive this curve to have three branches, two infinite and pofitive, and one negative, together with a *maximum*, which at prefent we can take no notice of; and the axis will be cut in two points. Another example.

Fig. 149.



Make $y = z + t$, whence we may have two equations, $x^4 = a^3z$, and $x^3 = aat$. To the axis AB let the parabola MAD of the equation $x^4 = a^3z$ be defcribed; and it being AF $= z$, it will be FD $=$ AE $= x$. Through the fame point A let the cubic parabola CAP of the equation $x^3 = aat$ be defcribed, and PE $= t$ will correfpond to the fame x. Whence, it being AE $= x$, it will be PE $+$ ED $= z + t = y$, making PD parallel to AF. Whence it may be feen, that, taking x pofitive, the ordinate y increafes *in infinitum*.

Fig. 150.



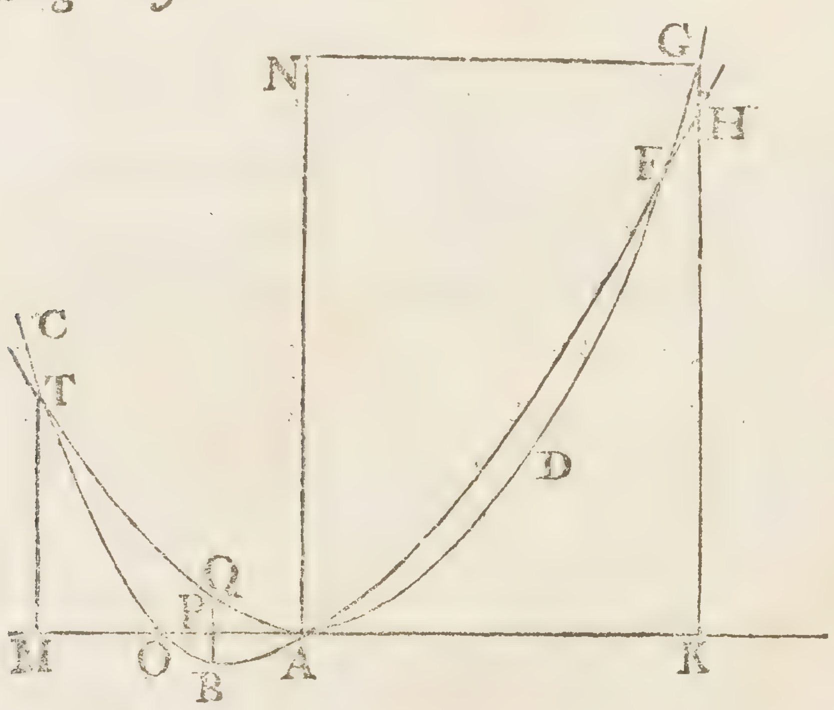
Then, taking x negative, t will be negative, and confequently $y = z - t$. Let AG $= x$ negative, it will be GM $= z$, GT $= t$, whence $y =$ MT negative; and among all the values of MT, there will be a greateft. Taking $x = -a$, it will be GM $=$ GT, whence $y = 0$. Taking x negative, and greater than a, it will be GM $-$ GT, a pofitive quantity; whence y will be pofitive, and will increafe *ad infinitum*. The curve will be nearly of the form of Fig. 150, taking x from the point A.

EXAMPLE III.

259. Let it be propofed to conftruct the curve of the equation $x^4 + ax^3 - aax^2 = a^3y$. This curve will have four branches, two pofitive and infinite, two negative and finite. It will cut the axis in two points, and will touch it in Another example to the first cafe.

one. It will have two negative *maxima*, &c. as will be known by the rules to be delivered in their due place.

Fig. 151.



Put $y = z - q$, and make the two equations $x^4 + ax^3 = a^3z$, and $-xx = -aq$. The curve of the equation $x^4 + ax^3 = a^3z$ we know already how to construct by help of this method, and let it be CBADG (Fig. 151.); in which, taking AK $= x$ positive, it will be KG $= z$. Taking x negative $=$ AP, it will be z negative $=$ PB; taking x negative and greater than AO, it will be z positive. To the axis AN let the parabola TAH of the equation $xx = aq$ be described. It being then AK $= x$ positive, it will be KH $= q$, and GH $= z - q = y$, which will increase *in infinitum* as x increases *in infinitum*. In the point F it will be $z = q$, and $y = 0$. Between the points F and A, q will be greater than z; whence $z - q$ will be a negative quantity, and y negative, and there will be a negative *maximum*. In the point A, it will be $z = 0$, $q = 0$, $y = 0$. Taking x negative equal to AP, it will be $z =$ BP, and negative; whence y is always negative. Between the points A and O there will be a *maximum* BQ; whence there will be a greatest q negative. Taking x negative and greater than AO, z will be positive, but less than q; whence y is negative. Taking x negative and equal to AM, it will be $z = q$, and $y = 0$. Taking x negative and greater than AM, it will be always z greater than q; whence it will be always y positive *in infinitum*.

If the equation should more abound in terms, the same artifice might be used; and, though the construction in this case might become more compounded and perplexed, yet, however, the method would still obtain.

We might construct the last equation in a different manner, by making $y = z + t - q$, and thence deriving three equations, $x^4 = a^3z$, $x^3 = aat$, $-xx = -aq$, and, by means of these three auxiliary curves, we might proceed to the construction of the principal curve; but I omit this for brevity.

The co-ordinates may make any angle.

260. Perhaps, in these constructions, and in the few that follow, it may seem necessary that the angle of the co-ordinates should be a right angle, it being always supposed to be such. But it will appear, after a little reflection, that this angle may be as we please; especially if we give a little attention to the angle of the co-ordinates of the subsidiary curves introduced, relatively to the angle of the co-ordinates of the curve of the given equation.

CASE

CASE II. EXAMPLE IV.

261. Let it be propofed to conftruct this equation, $x^n \pm a^s x^{n-s} \pm a^m x^{n-m}$, &c. $= y^t$. Make $y^t = a^{t-1} z$, and fubftituting this value inftead of y^t, the equation will be $x^n \pm a^s x^{n-s} \pm a^m x^{n-m}$, &c. $= a^{t-1} z$. By the method of the firft cafe, this curve may be conftructed; then defcribe the parabola of the equation $y^t = a^{t-1} z$, and we fhall have the relation between x and y in the propofed equation.

The fecond cafe of curves conftructed.

CASE III. EXAMPLE V.

262. Let it be propofed to conftruct the equation $x^m \pm ax^n \pm bx^s$, &c. $= y^p \pm y^q$, &c. Make $y^p \pm y^q$, &c. $= z$; then, by fubftitution, the equation will be $x^m \pm ax^n \pm bx^s$, &c. $= z$. By the method of the firft cafe, each of thefe two auxiliary curves may be conftructed to the fame axis, in which z is to be taken; and we fhall have the relation of the two co-ordinates x and y of the curve propofed.

The third cafe conftructed, with a general example.

263. Hitherto I have confidered only thofe equations which have their indeterminates feparate; fo that, when the indeterminates are involved with each other, the rules hitherto given cannot take place.

To feparate the indeterminates when involved.

In thefe cafes there is need, either by the common divifion, or by the extraction of roots, or by a congruous fubftitution, or by other expedients, to contrive a feparation of the faid indeterminates. As, if we had the equation $a^3y + ax^2y = a^2x^2 + x^4$, dividing by $a^3 + ax^2$, it would be $y = \frac{aaxx + x^4}{a^3 + axx}$. And, if the equation were $aaxy + xxyy = x^4 + a^4$, making the fubftitution of $z = \frac{yx}{a}$, we fhould have the equation $a^3z + aazz = x^4 + a^4$, in which the indeterminates or unknown quantities are feparate.

The propofed equations being thus prepared, we may proceed to their conftruction in the following manner.

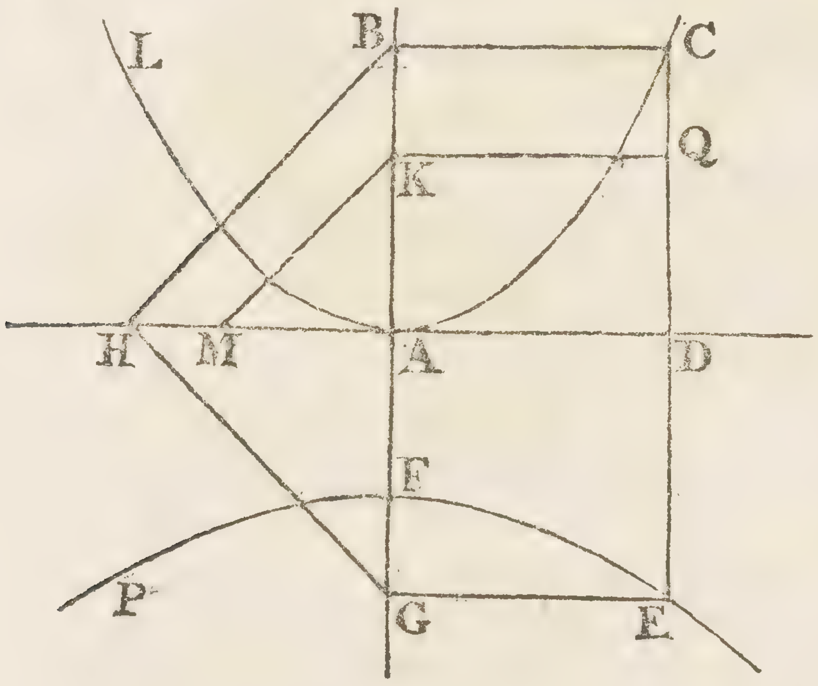
EXAMPLE VI.

Example of the construction of these *loci*.

264. Let the equation to be constructed be $y = \frac{a^2x^2 + x^4}{a^3 + ax^2}$. Make $a^2x^2 + x^4 = a^3p$. Make also $a^3 + ax^2 = a^2t$; and substituting these values in the equation proposed, it will be $y = \frac{ap}{t}$, that is, $t \,.\, p :: a \,.\, y$.

The proposed curve will have two branches, which stretch out *ad infinitum*. Positive y will correspond to x either positive or negative.

Fig. 152.



To axis HD let the curve LAC of the equation $aaxx + x^4 = a^3p$ be described; and, taking AD $= x$, it will be DC $= p$ $=$ AB. Take AF $= a =$ AM, then with vertex F, to the axis HD, let the curve PFE of the equation $a^3 + ax^2 = a^2t$ be described; and, taking AD $= x$, it will be DE $= t$. Whence, it being DC $= p$, and DE $= t$, draw EG parallel to AD, and from the point G draw GH at half a right angle, and it will be AH $= t$. From the point C draw CB parallel to DA, and draw the line BH, to which let MK be parallel. It being AD $= x$, it will be AK $= y$; for, because of similar triangles AMK, AHB, it will be AH . AB :: AM . AK; that is, $t \,.\, p :: a \,.\, \text{AK} = \frac{ap}{t} = y$. Whence, drawing KQ parallel to the axis, the lines AD, DQ, will be the two co-ordinates of the curve proposed. To obtain the other branch of our curve, it will suffice to take x on the negative side, and to repeat the same construction on the contrary part.

EXAMPLE VII.

Another *locus* constructed.

265. Now let it be proposed to construct the other equation $a^2xy + x^2y^2 = x^4 + a^4$, which, being managed by the rules for affected quadratick equations, may have the indeterminates separated. Or, by the substitution of $z = \frac{xy}{a}$, it will be reduced to $a^3z + aazz = x^4 + a^4$. This equation may be constructed

ſtructed by the method of the third caſe, and we ſhall have the two co-ordinates x and z. Then make the analogy, $x \,.\, z :: a \,.\, y$, which will be the ordinate required. If one ſubſtitution be not enough, to free the indeterminates from being involved together, we muſt try more than one; and when none will ſucceed, the equations elude this method, and we muſt have recourſe to other artifices.

266. A convenient ſubſtitution may alſo be of uſe in other caſes, in which the indeterminates are already ſeparate; and may often ſuggeſt a conſtruction which is more eaſy and elegant. Wherefore it may not be amiſs to try ſeveral ways, that we may chooſe that which will prove to beſt advantage. An obſervation.

EXAMPLE VIII.

267. Let the equation be $y^4 - 4ay^3 + 4aayy = 2a^3x$. Make $2a^3x = z^4$, and therefore it will be $y^4 - 4ay^3 + 4aayy = z^4$, that is, $yy - 2ay = zz$, or $2ay - yy = zz$. Concluſion of the examples.

Therefore I conſtruct this *locus*, which in the firſt caſe will be, by two oppoſite equilateral hyperbolas, with tranſverſe axis equal to $2a$; and in the ſecond caſe, by a circle with diameter $= 2a$: and, in general, by this and that together.

Fig. 153.

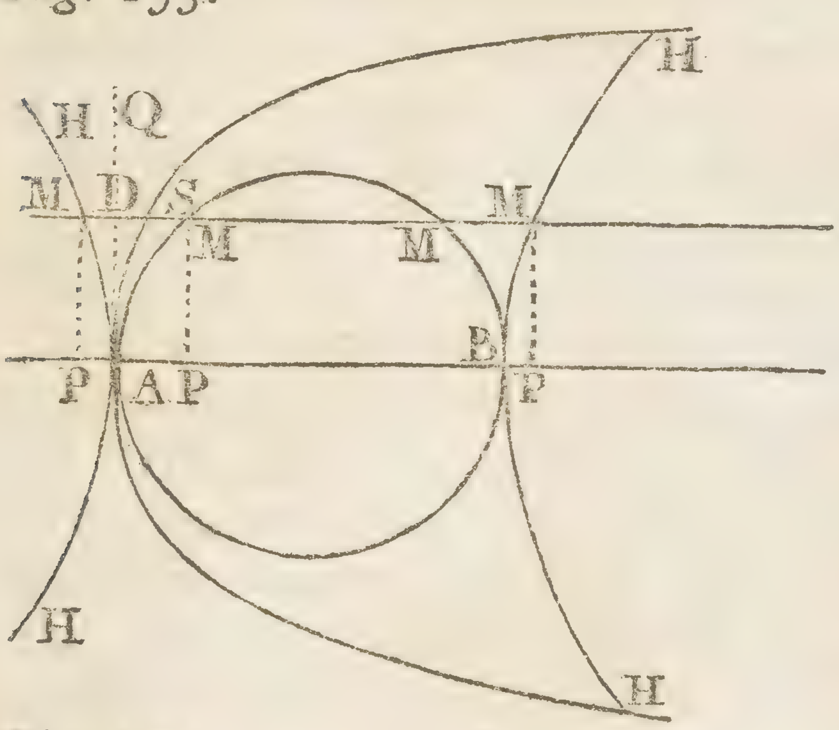
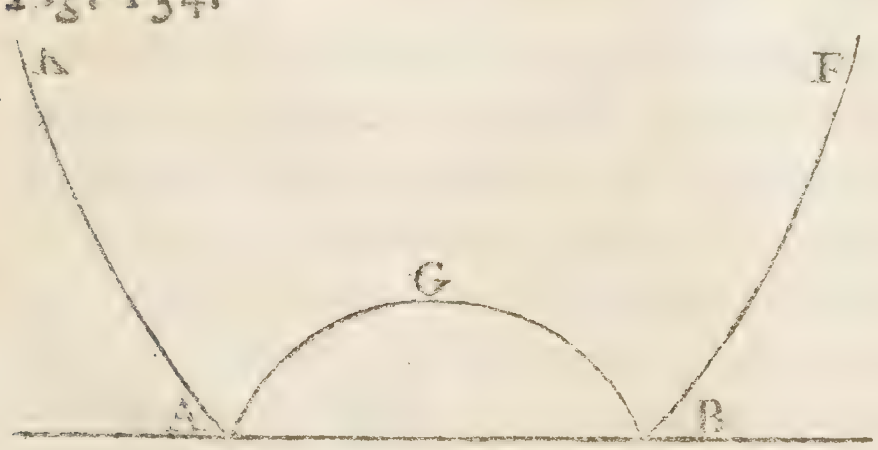


Fig. 154.



With tranſverſe diameter AB $= 2a$, (Fig. 153.) let there be deſcribed the two equilateral hyperbolas AMH, BMH, and the circle AMB. Then with vertex A, let the parabola of the equation $2a^3x = z^4$ be deſcribed, and raiſing the indefinite perpendicular AQ, and taking any line AD $= z$; then drawing MM parallel to AB, it will be DS $= x$, and DM $= y$, poſitive in the circle and in the hyperbola from A towards B, and negative in the hyperbola on the oppoſite part; and the curve will be nearly as KAGBF (Fig. 154.); in which the two branches, BF poſitive and AK negative, will go on *ad infinitum*; and there will be no branch under the axis AB, becauſe it can never be x negative.

SECT. VI.

Of the Method De Maximis et Minimis, *of the Tangents of Curves, of Contrary Flexure and Regreſſion; making uſe only of the Common* Algebra.

To find the *maxima* and *minima* of quantities, by compariſon with an equation of two equal roots.

268. Although the Calculus of Infiniteſimals be the ſimpleſt and the ſhorteſt method, and alſo the moſt univerſal, for managing ſuch ſpeculations; yet I was willing, before I finiſhed this Tract of Analyticks, or of what is called the *Carteſian* or Common *Algebra,* to ſhow very briefly, and by way of introduction, how the ſolution of ſuch queſtions may be performed, in geometrical curves, or ſuch as are expreſſed by finite algebräical equations, without the aſſiſtance of the *Differential Calculus,* or what is alſo called *The Method of Fluxions.*

Fig. 155.

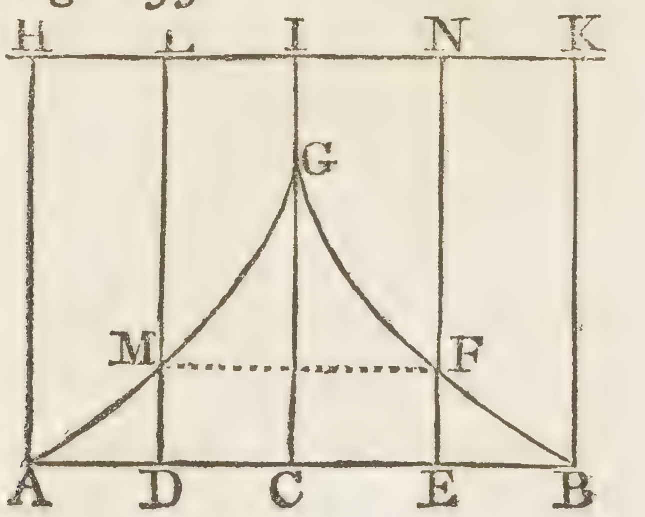
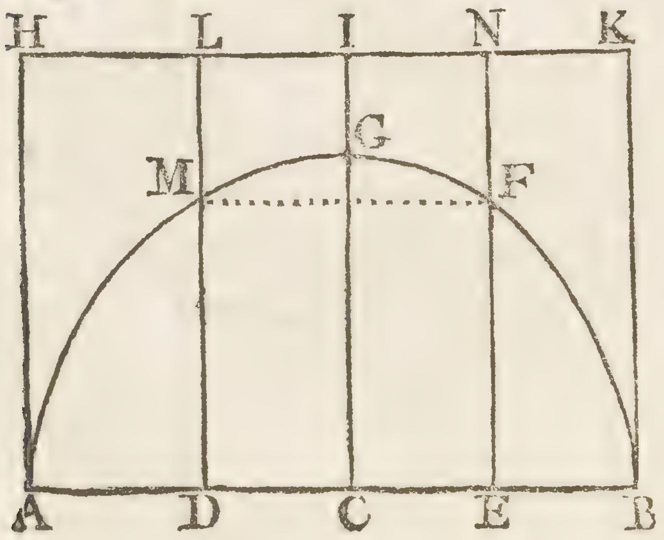


Fig. 156.



And to begin by the *Maxima* and *Minima*; that is to ſay, to find in geometrical curves the greateſt or the leaſt ordinates. Let the curve be AGB (Fig. 155, 156.), and taking any ordinate DM, draw MF parallel to the axis of the abſciſſes AB, the two ordinates DM, EF, will be equal, to which two different abſciſſes AD, AE, will correſpond. But the more the ordinates DM, EF, ſhall move approaching nearer to each other, the difference of the abſciſſes AD, AE, ſhall be ſo much the leſs; till at laſt the two ordinates DM, EF, coinciding with the greateſt ordinate CG, or the two LM, NF, with the leaſt IG, the abſciſſes AD, AE, or HL, HN, ſhall become equal in reſpect of the axis HK. Therefore, when the ordinate is the greateſt or the leaſt, the equation of the curve, diſpoſed according to the letter which expreſſes the abſciſs, ought to have two equal roots. To determine which, there is to be formed an equation of two equal roots, for example, $xx - 2ex + ee = 0$, which is the product of $x - e$ into $x - e$; and let the curve whoſe

whofe greateft or leaft ordinates are required, be the ellipfis $xx - 2ax + \frac{2ayy}{p} = 0$, for example, the abfciffes being taken from the vertex. Let this equation be compared, term by term, with the equation formed from two equal roots, in the following manner:

$$xx - 2ax + \frac{2ayy}{p} = 0.$$
$$xx - 2ex + ee = 0.$$

From the comparifon of the fecond terms, we find $a = e$; but e is the root of the equation $xx - 2ex + ee = 0$, and therefore $e = x$, and alfo $a = x$; and becaufe x is already determined, the comparifon of the laft terms will be fuperfluous. Wherefore, taking $x = a$, the correfponding ordinate in the ellipfis will be the greateft, as is already known, it being then half the conjugate axis.

But if the equation of the curve had been of the third, fourth, or higher degree, that we might make the comparifon, it would be neceffary that the equation of two equal roots, $xx - 2ex + ee = 0$, fhould be reduced to the fame degree as is the equation propofed, by multiplying it by fo many roots, whatever they may be, as there may be occafion for. Let the curve belong to this equation of the third degree, $x^3 * - axy + y^3 = 0$, (the afterifk * is put in the place of the fecond term which is wanting, and which fhould always be done, as often as any term is abfent,) of which we require the greateft ordinate. Therefore I multiply the equation $xx - 2ex + ee = 0$ by $x - f = 0$, and compare the product with the equation propofed,

$$\begin{array}{lllll} x^3 & * & - axy & + y^3 & = 0. \\ x^3 & - 2ex^2 & + eex & - eef & = 0. \\ & - fx^2 & + 2efx & & \end{array}$$

From the comparifon of the fecond terms, I find $-2e - f = 0$, and therefore $f = -2e$. From the comparifon of the third, I find $2ef + ee = -ay$, and fubftituting the value of f, it is $-3ee = -ay$. But $e = x$, therefore $y = \frac{3xx}{a}$. Inftead of y, if we fubftitute this value in the equation of the curve, it will give us $x = \frac{\sqrt[3]{2a^3}}{3}$, to which correfponds the greateft ordinate y, which will be $\frac{a \times 2^{\frac{2}{3}}}{3}$, or $\frac{\sqrt[3]{4a^3}}{3}$.

269. But, without comparing the given equation with another, which contains two equal roots, to fatisfy the condition of the Problem, it will be fufficient to multiply it, term by term, by any arithmetical progreffion. For, if the equation has two equal roots, as it ought to have in the cafe of a *maximum* or *minimum*, one of thofe roots will alfo, of neceffity, be included in the product of that equation multiplied by the arithmetical progreffion. Whence, by thus multiplying the equation, the condition will be included, under which the value of

To find the fame by multiplying by an arithmetical progreffion.

of the abſciſs will be found, to which the greateſt or leaſt ordinate correſponds. Now, to demonſtrate this, let the equation of the two equal roots be in general this, $xx - 2bx + bb = 0$, which let be multiplied by the arithmetical progreſſion a, $a + b$, $a + 2b$, and the product will be

$$\begin{array}{r} axx - 2abx + abb = 0. \\ - 2bbx + 2bbb \end{array}$$

In this ſubſtitute the quantity b inſtead of x, and all the terms will deſtroy one another. Or elſe, dividing it by $x - b$, the diviſion will ſucceed. Therefore $x - b$ will be one root of that product, as it is of $xx - 2bx + bb = 0$. The ſame will obtain if the arithmetical progreſſion be decreaſing, as a, $a - b$, $a - 2b$, $a - 3b$, &c.

Now, becauſe the equation of the two equal roots is general, and the arithmetical progreſſion a, $a + b$, $a + 2b$, &c. is general alſo, it will always be true, that when an equation of two equal roots is multiplied, term by term, by any arithmetical progreſſion, the product will be diviſible by one of thoſe roots. For the ſame reaſon, if an equation ſhall have three equal roots, and be multiplied by an arithmetical progreſſion, the product will have two of thoſe equal roots. And if this product be multiplied again by an arithmetical progreſſion, the new product will have one of thoſe roots. And ſo we may go on to ſuperior equations.

I reſume the equation to the ellipſis $xx - 2ax + \frac{2ayy}{p} = 0$, which I multiply by the progreſſion 2, 1, 0.

$$\begin{array}{cccc} xx & - 2ax & + \frac{2ayy}{p} & = 0. \\ 2, & 1, & 0. & \end{array}$$

The product is $2xx - 2ax = 0$, which gives $x = a$, as is found above. I multiply the ſame equation by another arithmetical progreſſion, 3, 2, 1,

$$\begin{array}{cccc} xx & - 2ax & + \frac{2ayy}{p} & = 0 \\ 3, & 2, & 1, & \end{array}$$

The product is $3xx - 4ax + \frac{2ayy}{p} = 0$, in which, inſtead of yy, I ſubſtitute it's value, $\overline{2ax - xx} \times \frac{p}{2a}$, given from the equation of the curve, and find $x = a$, as before.

I take the ſecond equation above, $x^3 * - axy + y^3 = 0$, and multiply it by the progreſſion 3, 2, 1, 0,

$$\begin{array}{ccccc} x^3 & * & - axy & + y^3 & = 0, \\ 3, & 2, & 1, & 0, & \end{array}$$

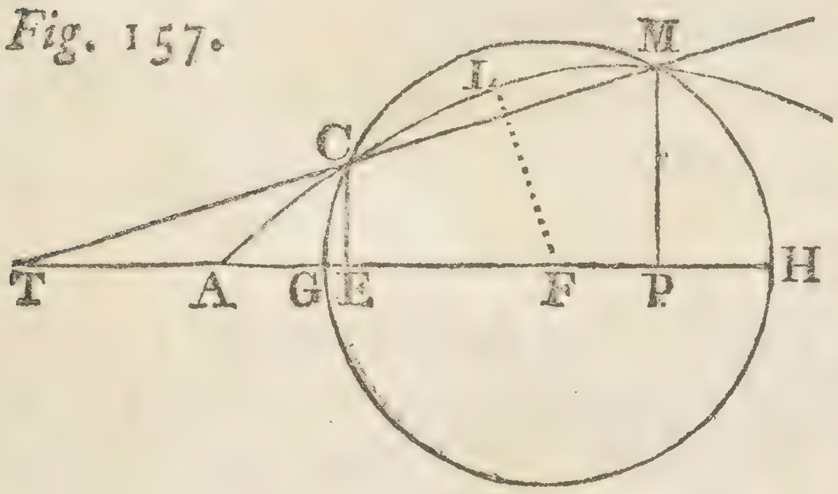
The product is $3x^3 - axy = 0$, or $3x^2 = ay$, as before.

270. By

270. By a like method may be found the tangents and perpendiculars to curves in any given points. Tangents and perpendiculars, how found.

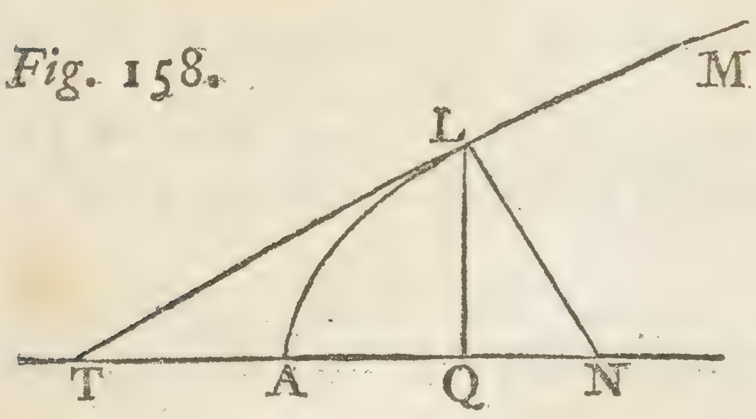
The queſtion is reduced to this; to find a circle that ſhall touch the curve in this point. For, in this caſe, the tangent of the circle in this point, as alſo the perpendicular or radius, will be in common to the curve alſo in the ſame point.

Fig. 157.



Let the curve be ACM, of which we deſire the tangent at the point L; and let the circle be GMH, which cuts it in the two points M, C. Drawing the two ordinates CE, MP, and the right line MCT, through the points M, C, it will cut the curve alſo in the points M, C. But the nearer theſe points ſhall approach to each other, the leſs always will be the difference of the ordinates CE, MP, and alſo of the abſciſſes AE, AP; ſo that when the two points coincide, for example at L, they will make the values equal of theſe ordinates, or of theſe abſciſſes; and then the circle will touch the curve in the point L. (Except when the curve and the circle are of equal curvature; for, in this caſe, the circle will both cut and touch the curve in the ſame point, as will be ſeen in the Differential Calculus.) The right line MT ſhall be a tangent both to the curve and the circle in the ſame point L; as alſo, FL will be a common perpendicular.

Fig. 158.



Therefore, in the curve ALM, make $AQ = x$, $QL = y$, and from the given point L drawing the right line LN, which we ſuppoſe to be perpendicular to the curve, and conſequently to the tangent at L; make $LN = s$, $AN = u$, and it will be $QN = u - x$. Then the right-angled triangle QLN will give the canonical equation $ss = uu - 2ux + xx + yy$, from which we are to have the value of y, or of x, and to ſubſtitute it in the equation of the given curve; by means of which we muſt have the value of s, or of u, conſidering x or y as given, becauſe we aſſume the point L as given.

Let the curve ALM, for example, be the *Apollonian* parabola of the equation $ax = yy$. Inſtead of yy, make a ſubſtitution of it's value given by the canonical equation, and we ſhall have $ax = ss - uu + 2ux - xx$; which being ordered according to the letter x, will be $\begin{matrix} xx - 2ux + uu = 0. \\ + ax - ss \end{matrix}$ This equation, therefore, ought to have two equal roots when the right line $LN = s$ is perpendicular

to

to the parabola in the point L, that is, in the cafe of a tangent. Therefore, the value of the indeterminate AN $= u$ being found, on the hypothefis of two equal roots, we fhall have the point N, from whence drawing NL to the given point L, and LT perpendicular to NL, that fhall be the tangent required.

Now, to determine the unknown quantity u on the fuppofition of two equal roots; I compare the equation, term by term, with one of two equal roots, that is, with $xx - 2ex + ee = 0$, after the following manner:

$$\left.\begin{array}{l} xx - 2ux + uu \\ \quad + ax - ss \end{array}\right\} = 0.$$

$$xx - 2ex + ee = 0.$$

Now, from the comparifon of the fecond terms, we fhall have $-2u + a = -2e$, or $u = \frac{1}{2}a + e$. But $e = x$, by the equation $xx - 2ex + ee = 0$. Therefore $u = \frac{1}{2}a + x$. Wherefore, from the point Q, taking QN $= \frac{1}{2}a$, NL will be the perpendicular, and LT, perpendicular to it, will be the tangent to the curve in the point L.

Inftead of comparing the faid equation with one of two equal roots, it may be multiplied by this arithmetical progreffion 3, 2, 1, thus:

$$\left.\begin{array}{l} xx - 2ux + uu \\ \quad + ax - ss \end{array}\right\} = 0.$$

$$3, \quad 2, \quad 1,$$

The product is $\left.\begin{array}{l} 3xx - 4ux + uu \\ \quad + 2ax - ss \end{array}\right\} = 0.$ But $ss = uu - 2ux + xx + yy$; and, by the parabola, it is $yy = ax$; whence $ss = uu - 2ux + xx + ax$. Subftituting, therefore, this value inftead of ss, it will be $2xx - 2ux + ax = 0$. That is, $u = \frac{1}{2}a + x$, as before.

We might have had our defire more compendioufly, by multiplying the equation by this arithmetical progreffion, 2, 1, 0.

Example. 271. Let the curve be the fecond cubical parabola $x^3 = ayy$. Making the fubftitution of the value of yy, derived from the canonical equation, there arifes the equation $\begin{array}{l} x^3 + ax^2 - 2aux + auu \\ \qquad\qquad\qquad\qquad - ass \end{array} = 0$, which, becaufe it is of the third degree, muft be compared with the product of the equation $xx - 2ex + ee = 0$ into $x - f = 0$; thus, $\left.\begin{array}{l} x^3 + ax^2 - 2aux + auu \\ \qquad\qquad\qquad\qquad - ass \end{array}\right\} = 0.$

$$\begin{array}{l} x^3 - 2ex^2 + eex - eef = 0. \\ \quad\; - fx^2 + 2efx \end{array}$$

By

By comparing the ſecond terms, we have $-2e - f = a$, that is, $f = -a - 2e$. From the compariſon of the third, it is $ee + 2ef = -2au$; and putting the value of f now found, it is $u = \frac{3ee + 2ae}{2a}$, that is, $u = \frac{3xx + 2ax}{2a}$, becauſe $e = x$.

Now I ſhall multiply the equation by the arithmetical progreſſion 3, 2, 1, 0,

$$\left.\begin{array}{llll} x^3 + ax^2 - 2aux + auu \\ \qquad\qquad\qquad\quad - ass \end{array}\right\} = 0,$$

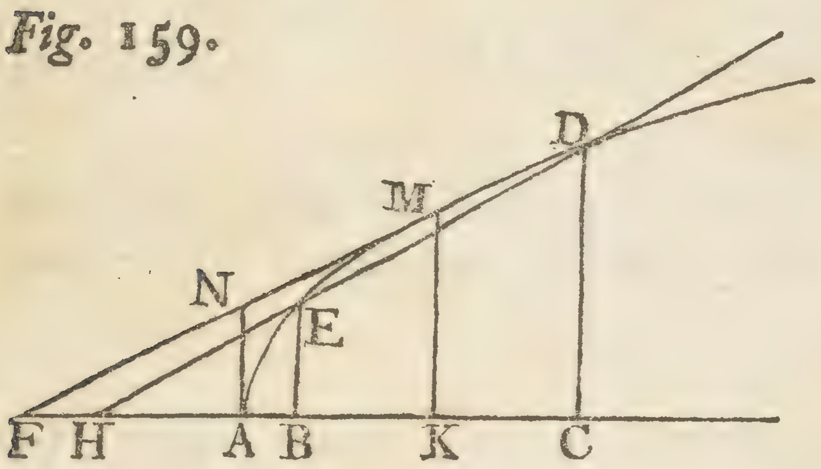
$$3, \quad 2, \quad 1, \quad 0,$$

The product is $3x^3 + 2ax^2 - 2aux = 0$, and therefore, in like manner, $u = \frac{3xx + 2ax}{2a}$.

272. Concerning the choice of a proper arithmetical progreſſion, it may be obſerved, that, generally, that will be the moſt convenient, which forms the exponents, beginning with the greateſt index of that letter according to which the equation is ordered. How to chooſe a progreſſion.

273. Another manner of ſolving this Problem may be this, which is ſomething different, but perhaps more ſimple, and which will be of uſe in contrary flexures and regreſſions. This Problem ſolved another way.

Fig. 159.

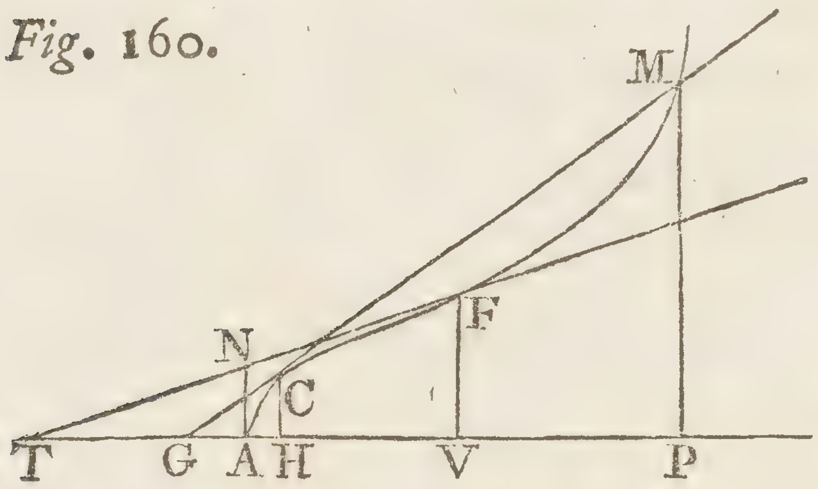


Let the curve AEMD be cut by the right line HED in the points E, D; and make the abſciſſes AB or AC $= x$, the ordinates BE or CD $= y$. It is plain that the right line HD going on to be the tangent FM of the curve in the point M, the two points E, D, will coincide in M, and conſequently will make the two lines AB, AC, equal to each other, as alſo the two lines BE, CD. Draw AN parallel to the ordinates, and make AF $= u$, AN $= s$. By the ſimilar triangles FAN, FKM, it will be $u \,.\, s :: u + x \,.\, y$; that is, $y = \frac{us + sx}{u}$, and $x = \frac{uy - us}{s}$. In the equation of the given curve, ſubſtitute theſe values inſtead of y or x, and another equation will ariſe from hence, which will have two equal roots, ſince AF, AN, are ſuch, as that the right line FNM touches the curve. Therefore, making a compariſon with another of two equal roots, or multiplying it by an arithmetical progreſſion, we ſhall have the value of AF or AN required; and one being given, the other will alſo be given. I forbear Examples, becauſe the manner of operation is the ſame as that uſed before.

274. As the nature of *maxima* and *minima*, and likewiſe of tangents, neceſſarily requires equations of two equal roots, ſo, in contrary flexures and regreſſions, Points of contrary flexure and regreſſion what, and how found.

greffions, three equal roots are required. By contrary flexure is meant that point, in which from concave the curve becomes convex, or the contrary; and by regreffion is meant that point in which the curve turns directly back again, whether concave or convex.

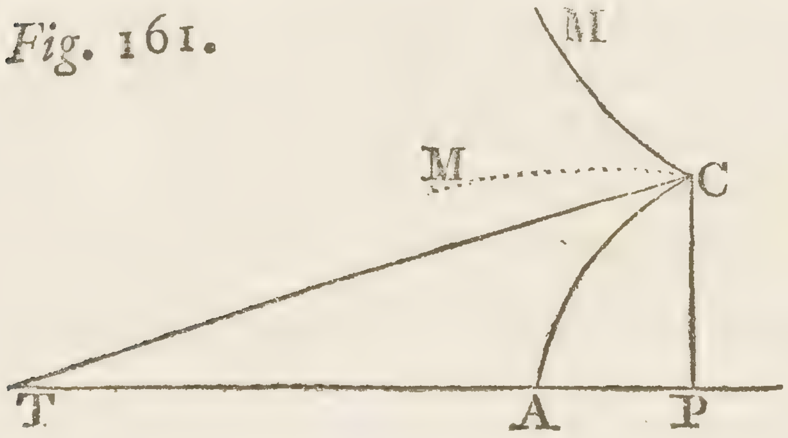
Fig. 160.



Let the curve be ACFM, which has a contrary flexure in the point F, and let be drawn the right line GCM, which touches it in the point C, and cuts it in the point M; from which draw the ordinates CH, MP. It is eafy to perceive, that the more the point C of the tangent fhall approach to the point F of contrary flexure, fo much the more alfo the point M fhall approach to the point F; fo that when the point C falls in with F, the point M will alfo fall in with it; and confequently AH, AP, will become equal, as alfo CH, MP, and the right line GCM will both touch and cut the curve in the point F. But the nature of the tangent already requires two equal roots, and now they are joined by a third; fo that the property of contrary flexure is fuch, that three equal roots are correfponding to it.

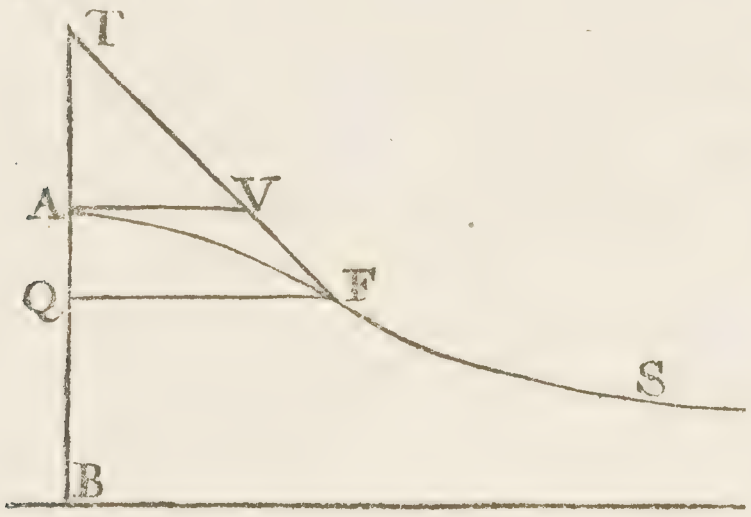
From the point A drawing AN parallel to the ordinates, and making AN $= s$, AT $= u$, and drawing TNF; becaufe of fimilar triangles TAN, TVF, it will be $y = \frac{us + sx}{u}$, and $x = \frac{uy - us}{s}$, making VA $= x$, and VF $= y$. Wherefore, fubftituting thefe values of x or y in the equation of the given curve, the equation that arifes ought to have three equal roots, when AT or AN are fuch that TNF, drawn from the point T through the point N, may meet the curve in F, the point of contrary flexure required.

Fig. 161.



In like manner we may reafon about the curve ACM, which has a regreffion in the point C. For the tangent TC of the curve in the point C, will alfo cut it in the fame point; and thence the three equal roots will arife after the fame manner.

Fig. 162.



Let AFS be the curve of the equation $ayy - xyy - aax = 0$, in which are AQ $= x$, and QF $= y$; and let the point F of contrary flexure be required. Make AT $= u$, AV $= s$, and QF parallel to the ordinates. Now, inftead of x, fubftituting it's value $\frac{uy - us}{s}$, in the equation of the curve, it will be

$$\left.\begin{array}{l} y^3 - \frac{asyy}{u} + aayy - aas \\ - syy \end{array}\right\} = 0.$$

This equation ought to have three equal roots, and therefore we muſt compare it with an equation of three equal roots; or elſe multiply it by two arithmetical progreſſions.

Let us multiply it, therefore, by the progreſſion 1, 0, − 1, − 2, and the product will be $y^3 * - aay + 2a^2s = 0$. Multiply it again by the progreſſion 3, 2, 1, 0, which will give us $3y^3 - aay = 0$, and therefore $yy = \frac{1}{3}aa$. This value, being ſubſtituted in the equation of the given curve, will laſtly produce $x = \frac{1}{4}a$.

275. The manner is the ſame for finding the regreſſions of curves, and this method is applicable to both. So that, to diſtinguiſh them, there is no other way, but to find, by means of a conſtruction, the figure and proceeding of the curve.

To diſtinguiſh contrary flexures from regreſſions, and *maxima* from *minima.*

The ſame ambiguity ariſes in queſtions *de maximis et minimis*, which only can be removed by acquiring ſome knowledge of the diſpoſition of the curve. By the ſame condition of three equal roots we may find the *Radii* of Curvature; but as I ſhall further treat of ſuch things in the following Volume, not to be too tedious, I ſhall here put an end to this.

END OF THE FIRST VOLUME.

I vol: XXVII, XLVI, 1 f.; 251 pp. - [manca il primo foglio]
figure nel testo

II vol: 2 ff., 371 pp. - FM